34th EPS Conference on Plasma Phys. Warsaw, 2 - 6 July 2007 ECA Vol.**31F**, P-5.061 (2007)

Asymptotic description of high frequency waves in tokamak

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1. Introduction.

As well known the linear properties of plasma waves are fully described by the constitutive relation that expresses plasma reaction to small fluctuations. In steady state, the relation links the electric field $\mathbf{E}$ with the displacement $\mathbf{D}$ as

$$\mathbf{D}(\mathbf{r}) = (2\pi)^{-3}\int \hat{\varepsilon}(\mathbf{r},\mathbf{r}-\mathbf{r}')\mathbf{E}(\mathbf{r}')d\mathbf{r}' = \int d\mathbf{k}\,\mathbf{E}(\mathbf{k})\hat{\varepsilon}(\mathbf{r},\mathbf{k})e^{i\mathbf{kr}}. \tag{1}$$

In the approach of the geometrical optics the relation is usually understood in the local sense. It means that the plasma response to a plane wave perturbation is assumed to be also a plane wave. This is fully valid for a homogeneous medium where the plane waves are eigenfunctions. However, even for a slightly inhomogeneous medium the assumption always violates.

As pointed out in [1] the violation has far reaching consequences and can lead to loss or wrong description of many physics effects. For instance, the plane-wave-based consideration cannot properly describe non-Hermitian terms and, as a result, the conservation of energy. Moreover, the local approach complies with the approximation of the geometric optics but is not consistent with the more precise quasi-optics techniques that are gaining acceptance in tokamak applications [2-4]. The correct treatment should be based on the concept of the wave packets rather than the plane waves. In this contribution, the ideas of Ref.[1] are applied to the beam tracing equations [5,6] and a simple but self-consistent recipe for including the local dielectric tensor into the beam tracing equations is presented.

2. Differential constitutive relation.

A space vector $\mathbf{r}$ will be represented in an arbitrary Cartesian coordinates as $\mathbf{r} = \{X^{\alpha}\}$ and the wave vector as $\mathbf{k} = \{k_{\alpha}\}$. Also the tensor summation convention will be employed: a summation with respect to every repeated index is implied. We introduce a vector function $\bar{\mathbf{k}} = \bar{\mathbf{k}}(\mathbf{r}) = \{\bar{k}_{\alpha}\}$ to be defined later and make use of the Taylor expansion

$$\hat{\varepsilon}(\mathbf{r},\mathbf{k}) = \hat{\varepsilon}(\mathbf{r},\bar{\mathbf{k}}) + \frac{\partial\hat{\varepsilon}}{\partial k_{\alpha}}\bigg|_{\bar{\mathbf{k}}}(k_{\alpha}-\bar{k}_{\alpha}(\mathbf{r})) + \frac{1}{2}\,\frac{\partial^2\hat{\varepsilon}}{\partial k_{\alpha}\partial k_{\beta}}\bigg|_{\bar{\mathbf{k}}}(k_{\alpha}-\bar{k}_{\alpha}(\mathbf{r}))(k_{\beta}-\bar{k}_{\beta}(\mathbf{r})) + \ldots \tag{2}$$

where $\hat{\varepsilon}$ and all its derivatives on the right hand side are computed at $\mathbf{k} = \bar{\mathbf{k}}(\mathbf{r})$. This makes the dependences on $\mathbf{k}$ in (2) explicit and allows to calculate integrals in (1) thus obtaining the differential form of the constitutive relation

$$\mathbf{D}(\mathbf{r}) = \left[\hat{\varepsilon}(\mathbf{r},\bar{\mathbf{k}}) + \frac{\partial\hat{\varepsilon}}{\partial k_{\alpha}}\bigg|_{k_{\alpha}=\bar{k}_{\alpha}}\hat{D}_{\alpha} + \frac{1}{2}\,\frac{\partial^2\hat{\varepsilon}}{\partial k_{\alpha}\partial k_{\beta}}\bigg|_{k_{\alpha}=\bar{k}_{\alpha}}\hat{D}_{\alpha}\hat{D}_{\beta} + \ldots\right]\mathbf{E}(\mathbf{r}) \tag{3}$$

Here the differential translation operator $\hat{D}_{\alpha}$ is defined as $\hat{D}_{\alpha} = -\bar{k}_{\alpha}(\mathbf{r}) - i\partial/\partial X^{\alpha}$.

The only assumption has been employed so far that the Taylor series (2) converges and can be integrated. Formally, the relation (3) is local but it describes a non-locality through the derivatives with respect to k_α and X^α. It is clear that the convergence, and therefore the practical utility, of the series (3) depends on the properties of the wave field $\mathbf{E}$ and can be granted if $(\hat{D}_\alpha)^n \mathbf{E}$ decreases with n fast enough. In what follows, it will be shown that the requirement is fulfilled when the electric field $\mathbf{E}$ has a form of a Gaussian wave packet localized along its space trajectory.

3. Ordering.

Consider weakly inhomogeneous medium in the short wavelength limit. More precisely, we assume that the refractive and dispersive properties of the medium have the characteristic length of variation L that is large compared with $\lambda = c/\omega$ so that $\kappa = L\omega/c = L/\lambda \gg 1$ is a large parameter. Then our assumption of slow variation means

$$L\partial\hat{\varepsilon}/\partial X^\alpha = \partial\hat{\varepsilon}/\partial x^\alpha = \mathscr{O}(\kappa^0) = \mathscr{O}(1), \qquad (\omega/c)\partial\hat{\varepsilon}/\partial k_\alpha = \partial\hat{\varepsilon}/\partial N_\alpha = \mathscr{O}(1),$$

where the dimensional variables X^α and k_α are replaced with the dimensionless $x^\alpha = X^\alpha/L$ and $N_\alpha = ck_\alpha/\omega$ respectively. It is seen that the differential operators $\hat{D}_\alpha$ and derivations with respect to k_α are included in Eq. (3) in pairs so that we can replace $\hat{D}_\alpha\partial/\partial k_\alpha$ by $\Delta_\alpha\partial/\partial N_\alpha$ where $\Delta_\alpha = c/\omega\hat{D}_\alpha = -\bar{N}_\alpha(\mathbf{r}) - i\kappa^{-1}\partial/\partial x^\alpha$. We rewrite now Eq.(3) as

$$\mathbf{D}(\mathbf{r}) = \left[\hat{\varepsilon}(\mathbf{r},\mathbf{N}) + \frac{\partial\hat{\varepsilon}}{\partial N_\alpha}\Delta_\alpha + \frac{1}{2}\frac{\partial^2\hat{\varepsilon}}{\partial N_\alpha\partial N_\beta}\Delta_\alpha\Delta_\beta + \cdots\right]\Bigg|_{\mathbf{N}=\bar{\mathbf{N}}} \mathbf{E}(\mathbf{r}). \tag{4}$$

Assume that the electric field has the eikonal form $\mathbf{E}(\mathbf{r}) = \mathbf{A}(\mathbf{r})\exp\{i\kappa S(\mathbf{r})\}$. We employ the concept of the complex eikonal and seek $S(\mathbf{r})$ in the form

$$S(\mathbf{r}) = \bar{N}_\alpha(\bar{\mathbf{r}})\,(x^\alpha - \bar{x}^\alpha) + \tfrac{1}{2}\left(s_{\alpha\beta}(\bar{\mathbf{r}}) + i\phi_{\alpha\beta}(\bar{\mathbf{r}})\right)(x^\alpha - \bar{x}^\alpha)\,(x^\alpha - \bar{x}^\alpha) + \cdots \tag{5}$$

where a new quantity $\bar{\mathbf{r}} = \{\bar{x}^\alpha\}$ is introduced. Once the quadratic form $\phi_{\alpha\beta}(\bar{\mathbf{r}})$ is positive definite the electric field $\mathbf{E}$ has a form of wave packet exponentially decaying with increasing $|\mathbf{r} - \bar{\mathbf{r}}|$. We observe that

$$\Delta_\alpha\mathbf{E}(\mathbf{r}) = e^{i\kappa S}\left[(S_\alpha - \bar{N}_\alpha)\,\mathbf{A} + \kappa^{-1}\partial\mathbf{A}/\partial x^\alpha\right] = \mathscr{O}\left(\kappa^{-1/2}\right), \tag{6}$$

where $S_\alpha = \partial S/\partial x^\alpha$. The latter equality in Eq. (6) shows that $(\Delta_\alpha)^n\mathbf{E}$ has to be ordered as $\mathscr{O}(\kappa^{-n/2})$ and follows from the standard estimate [5].

Physically, the ordering means that the propagation properties of the wave packets, i.e. refraction, diffraction and absorption, are due to the small space domain where the wave field is localized. Remote regions contribute to the plasma response with exponentially decreasing weight. The estimate (6) shows that the omitted terms in Eqs. (4), (5) should be ordered as $\mathscr{O}(\kappa^{-3/2})$ and thus go beyond the accuracy of the asymptotic solution. On the other hand, straightforward replacing Eq.(1) with the local dispersion relation and taking into account only the first term in Eq.(4) can result in neglecting essential terms.

4. Dispersion relation for wave packets.

On substitution the eikonal ansatz and Eq. (4) into the Maxwell equations and repeating all calculations of Ref. [5] one arrives at the dispersion relation

$$H = \det \|\mathbf{kk} - k^2 \hat{\mathscr{I}} + (\omega/c)^2 \hat{\mathscr{E}}\| = 0 \tag{7}$$

where the only difference is that one has to replace the "cold" tensor $\hat{\varepsilon}(\mathbf{r})$ of Ref. [5] with the quantity

$$\hat{\mathscr{E}} = \hat{\varepsilon} + \left.\frac{\partial \hat{\varepsilon}}{\partial k_\gamma}\right|_{\mathbf{k}=\bar{\mathbf{k}}} (k_\gamma - \bar{k}_\gamma) + \frac{1}{2} \left.\frac{\partial^2 \hat{\varepsilon}}{\partial k_\gamma \partial k_\delta}\right|_{\mathbf{k}=\bar{\mathbf{k}}} (k_\gamma - \bar{k}_\gamma)\left(k_\delta - \bar{k}_\delta\right). \tag{8}$$

where the tensor $\hat{\varepsilon}(\mathbf{r},\mathbf{k})$ is introduced in Eq.(1) and includes the effects of the space dispersion. It remains to define the two vectors $\bar{\mathbf{r}} = \{\bar{x}^\alpha\}$ and $\bar{\mathbf{k}} = \{\bar{k}_\alpha\}$.

Up to this point, they were treated as free independent vector parameters in space and spectral space, respectively. We select now these parameters as solutions to the Hamiltonian set of equations

$$d\bar{x}^\alpha/d\tau = \partial H/\partial k_\alpha, \qquad d\bar{k}_\alpha/d\tau = -\partial H/\partial x^\alpha. \tag{9}$$

The equations (9) define the space curve, called the reference ray, $\bar{\mathbf{r}}(\tau) = \{\bar{x}^\alpha(\tau)\}$ and the vector function $\bar{\mathbf{k}}(\tau) = \{\bar{k}_\alpha(\tau)\}$ along this curve. Equation (5) shows that the wave field is localized in the close vicinty of the reference ray. Moreover, the spectrum is localized in the vicinity of the wave vector $\bar{\mathbf{k}}$. These properties are physics background for Taylor expansions (2) and (5). The definition of $\bar{\mathbf{r}}$ and $\bar{\mathbf{k}}$ completes the derivation of the dispersion relation and generalizes the beam tracing procedure to the media with the space dispersion.

The two functions $s_{\alpha\beta}$ and $\phi_{\alpha\beta}$ introduced in Eq.(5) describe the curvature of the wave front and the divergence of the wave packet, respectively. Equations for these quantities retain exactly the same form as in [5] except for H that is now defined by Eqs.(7), (8). All results of Ref. [5] can be word for word repeated with regard to the new dispersion function (7).

The new form of the dispersion relation (7) with the effective dielectric tensor $\hat{\mathscr{E}}$ is the main result of this contribution. It is worth to mention that exactly the same recipe is applicable to the electrostatic case. Namely, all constructions of Ref. [6] remain valid provided the dispersion relation of the plasma is written in the form $H = k_\alpha k_\beta \mathscr{E}^{\alpha\beta} = 0$ where $\mathscr{E}^{\alpha\beta}$ are contravariant components of the effective dielectric tensor $\hat{\mathscr{E}}$ defined by Eq. (8). It is also obvious that the limiting case of no space dispersion $\hat{\varepsilon} = \hat{\varepsilon}(\mathbf{r})$ follows immediately from (8) because the "cold" tensor has no $\mathbf{k}$ dependence.

5. Discussions and conclusions.

The beam tracing approach provides a solution to the Maxwell equations as an asymptotic expansion in descending powers of the large parameter $\sqrt{\kappa}$ up to the accuracy of $\mathscr{O}(\kappa^{-3/2})$. As already mentioned the quantity $-\kappa\phi_{\alpha\beta}(x^\alpha - \bar{x}^\alpha)(x^\beta - \bar{x}^\beta)$ in the exponent of $\mathbf{E}(\mathbf{r})$ ensures that the wave packet is strongly localized around $\bar{\mathbf{r}}$ in the configuration space and around $\bar{\mathbf{k}}$ in the spectral space. Because of this property only the close vicinity of the wave packet centre participates in the plasma response to the electric field and makes it possible to replace eventually the integral constitutive relation (1) with the algebraic dispersion relation (7). Derivation of Eq.(7) shows that within the accuracy of $\mathscr{O}(\kappa^{-3/2})$ one need not keep further terms of expansion in (2) and (8) that contribute to the higher orders of the asymptotic expansion. Moreover, the exact constitutive relation is not needed for the beam tracing technique (as well as for any other quasi-optics approach) because the solution does not support this level of accuracy.

As an illustration to the aforesaid consider the group velocity of the wave packet that follows from Eq. (9) and is proportional to the quantity $V^\alpha = \partial H/\partial k_\alpha$. In case of electrostatic waves one can write

$$V^\alpha = \frac{\partial H}{\partial k_\alpha}\bigg|_{\bar{k}_\alpha} = \left[\bar{k}_\nu\left(\varepsilon^{\alpha\nu} + \varepsilon^{\nu\alpha}\right) + \bar{k}_\nu \bar{k}_\mu \frac{\partial \varepsilon^{\nu\mu}}{\partial k_\alpha}\right]\bigg|_{\bar{k}_\alpha}. \tag{10}$$

The last term on the right hand side describes the influence of the space dispersion on the refraction exactly in the form that one would obtain without using the expansion (2).

This result is hardly surprising. It confirms that the approach of the geometric optics is intrinsically consistent: the ansatz and the result are concordant in the asymptotic ordering. Although the local dispersion relation of the geometrical optics, $\mathbf{D}(\mathbf{r},\mathbf{k}) = \hat{\varepsilon}(\mathbf{r},\mathbf{k})\mathbf{E}(\mathbf{r},\mathbf{k})$, is obtained as a plasma response to an infinite plane wave it is used to describe an infinitely thin ray. Nevertheless, there is no contradiction. In this respect, the two extreme concepts come together: the infinitely thin ray and the plane wave in a homogeneous medium do not "know" anything about non-locality of the medium. As a result, the geometrical optics has the adequate asymptotic accuracy. However, taking account of a finite ray width or associated with that non-corpuscular wave characteristics necessarily involves non-local properties of the dispersion. For this case, one can prove that the expression (8) is consistent in accuracy with the basic assumption (5) of the beam tracing description.

In conclusion, it is shown that the beam tracing technique can be straightforwardly extended to the media with the space dispersion. All derivations and formulas of [4-6] remain unchanged provided that the "cold" dielectric tensor $\hat{\varepsilon}(\mathbf{r})$ is replaced by $\hat{\mathscr{E}}(\mathbf{r},\mathbf{k})$ given by Eq.(8). Contribution to the effect of refraction is due to the second term in the expansion (8) while the third term controls the wave front deformation and the wave beam divergence. As known [1], absorption of waves is affected by the new terms even more strongly. This effect is described by the amplitude equation and will be discussed elsewhere. The results of this contribution are applicable to all existing quasi-optics techniques because they differ in equations and in the way of solution but are exactly the same in the asymptotic expansion.

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