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Linear tearing mode equations for low collisionality

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1. Introduction

Magnetic reconnection is believed to play a role in important tokamak phenomena such as sawteeth and disruptions. It is normally analysed using MHD or two-fluid Braginskii plasma models. However for large hot tokamaks like JET or ITER the collisionality is such as to place them in the banana regime where neoclassical effects intervene, leading to instability driven by the bootstrap current [1-3]. Furthermore, in cylindrical geometry the electron dynamics falls in the 'semi-collisional fluid' regime where tearing mode instability requires large Δ' [4]. Here we develop a theory of the resonant layer physics describing both these competing effects, and also include the effect of temperature gradients, which may modify the results of Refs. 2, 3. Interestingly there is experimental evidence [5], for example, that the sawtooth instability is triggered when the electron temperature gradient, dT_e/dr, exceeds a critical value. While our electron description is of wide relevance, the fluid treatment of the ions used requires the ion banana orbit width to be less than the semi-collisional electron layer. This limits the application of the present theory to low magnetic shear – however this is highly relevant to the sawtooth instability.

2. Derivation of eigenvalue equations

(i) Electron and ion gyro-kinetic solutions. We solve for the electron and ion responses, f_j, to perturbed electromagnetic fields: electrostatic potential, Φ, parallel vector potential $A_{\|} = \nabla_{\|}\Psi / i\omega$, and parallel component of the magnetic field, $\delta B_{\|}$ (e.g. $\Phi(x,\theta)\exp(in\varphi - im\theta - i\omega t)$ where x is a flux surface label and φ, θ are toroidal and poloidal angles in a straight field line co-ordinate system), using a gyro-kinetic model. In standard notation, with $f_j = -\left(e_j\Phi f_{0j}/T_j\right) + g_j e^{iL_j}$ [3],

$$\left(Iv_{\|}/R^2Bq\right)\left(\frac{\partial g_j}{\partial\theta} + inq'xg_j\right) + \mathbf{v}_{dj}.\nabla g_j - i\omega g_j - C_j(g_j)$$

j = e:	1	ε^2	ε	ε^3	ε
j = i:	1	ε^3	ε	ε^3	ε

$$= -\left(ie_j/T_j\right)f_{0j}\left(\omega - \omega_{*j}^T\right)\left[J_0(\Phi - v_{\|}A_{\|}) + J_1\left(v_\perp\delta B_{\|}/k_\perp\right)\right] \quad (1)$$

where we have indicated a possible ordering scheme for electrons and ions. This ensures we are in the banana regime for the application of neoclassical theory and the semi-collisional regime, $\omega \sim k_{\|}^2 v_{the}^2 / \nu_e$, where $k_{\|} = nq'x / Rq$, when parallel collisional diffusion for electrons balances the mode frequency. A more complex ordering scheme allows one to consistently include ion neoclassical cross-field transport, but we omit this here for simplicity. Parallel diffusion effects for ions and ion sound effects are ignored. The large geodesic (radial) component, v_{drj}, of the magnetic drift appears at $O(\varepsilon)$ in eqn.

(1) because of the narrow width of the resonant layer. The arguments, $\left(k_\perp v_\perp/\Omega_{cj}\right)$, of the Bessel functions, $J_{0,1}$, can be set to zero (but first order corrections for the ions are retained when eqn. (1) is used to derive the vorticity equation later) - the validity of this requires low magnetic shear, s; the normal magnetic curvature drift also plays a role in the vorticity equation. It is convenient to introduce h_j into eqn. (1), where $g_j = \left(e_j\Psi f_{0j}/T_j\right)\left(1-\omega_{*j}^T/\omega\right)+h_j$.

The solution of eqn. (1), order by order, mirrors that in standard neoclassical theory, with arbitrary functions being determined by collisional constraints arising from periodicity of the bounce/transit motion (we use momentum conserving, pitch-angle scattering collisional models). The lowest order solution for h_j is Maxwellian with perturbed densities, $\hat{n}_j$, and temperatures, $\hat{T}_j$, constant on a flux surface. In next order one determines perturbed parallel flows from which one can compute the ion flow and bootstrap current driven by the perturbed gradients. Finally, solubility conditions in third order lead to neoclassical 'fluid equations' for $\hat{n}_j$ and $\hat{T}_j$ in response to the electromagnetic perturbations, which balance cross-field neoclassical transport, parallel collisional electron transport and ω. In what follows we shall ignore the small electron cross-field transport, enabling us to obtain closed algebraic expressions for the semi-collisional perturbed electron density and temperature (we also ignore some finite banana width terms proportional to s << 1). These expressions are *rational* functions of the semi-collisionality factor, $Q \equiv i\left[\left(\mathrm{In}q'x\left\langle 1/R^2\right\rangle\right)^2/\left\langle B^2\right\rangle\right]\left(T_e/m_e\omega\nu_{ei}\right)\ \left(f_c/\left(1-0.37f_c\right)\right)$, where f_c is the fraction of passing particles and $\langle\ \rangle$ is a flux surface average. For instance we display the effect of Q on the parallel electric conductivity:

$$\sigma_\parallel^{sc} = \sigma_0 D_0^{-1}\left\{\left(1-\omega/\omega_{*e}\right)\left[2.45-0.45f_c+\left(15.9-7.59f_c+0.63f_c^2\right)Q\right]-\left(\omega\eta_e/\omega\right)\left[5.94-1.69f_c\right]\right\}$$
$$D_0 = 1+\left(18.55-6.05f_c\right)Q+\left(15.9-7.59f_c-0.63f_c^2\right)Q^2 \qquad (2)$$

The corresponding equations for the ion quantities $\hat{n}_i$ and $\hat{T}_i$ are linearly algebraic if we ignore ion neoclassical transport, otherwise they involve a second order ODE.

(ii) Perturbed field equations. To obtain equations for the perturbed fields we utilise quasi-neutrality and the parallel and perpendicular Ampère's equations for Ψ and $\delta B_\parallel$, using the perturbed currents calculated from the gyro-kinetic solutions. The perpendicular Ampère's law relates $\delta B_\parallel$ to the perturbed pressure, $\tilde{p}$; thus it is small in β and we shall neglect it apart from its usual role in converting ∇B drifts to curvature drift in the vorticity equation. The parallel Ampère's law in leading order implies $\Psi^{(0)}$ is flute-like and quasi-neutrality then implies the same for $\Phi^{(0)}$. Imposing the poloidal periodicity constraint as a solubility constraint in the next order of the parallel Ampère's law for $\Psi^{(1)}$ provides a first order ODE for $\Psi^{(0)}$ in terms of the flux surface average of the neoclassical parallel current. This equation serves as an effective Ohm's law relating $\Phi^{(0)}$ and $\Psi^{(0)}$ (when $\hat{n}_j$ and $\hat{T}_j$, j = e, i have been solved in terms of $\Phi^{(0)}$ and $\Psi^{(0)}$). It takes the form:

$$-\left(nq'I/\omega q\right)d^2\left(x\Psi^{(0)}\right)\Big/dx^2 = \left\langle j_{\parallel}B\Big/\left|\nabla\psi\right|^2\right\rangle \qquad (3)$$

where

$$\left\langle j_{\parallel}B\Big/\left|\nabla\psi\right|^2\right\rangle =$$

$$\left(q'/p_0'V'\right)\left\{\left(inq'L\sigma_{\parallel}^{sc}\left(Q\left(x\right)\right)\Big/\left\langle 1/R^2\right\rangle\right)x\left(\Phi^{(0)}-\Psi^{(0)}\right)+H\tilde{p}'+L\left(1-f_c\right)\left[\alpha_n\tilde{n}_e'\left(T_e+T_i\right)+\alpha_e n\tilde{T}_e'+\alpha_i n\tilde{T}_i'\right]\right\} \qquad (4)$$

with the three terms representing the inductive (with neoclassical/semi-collisional conductivity), Pfirsch-Schlüter and bootstrap current contributions, respectively. Here the flux-surface averaged equilibrium quantity L is defined in Refs.2, 3 and H in Ref. 6, the α_k are numerical coefficients involving f_c; and $\tilde{n}_j = \int d^3v f_j$ etc.

(iii) The vorticity equation. To close the system of equations we use the vorticity equation in the long wavelength limit of the ion FLR Bessel functions [3]. In lowest order we confirm $\Psi^{(0)}$ is flute-like and in first order we obtain an equation for $\Psi^{(1)}$; this can be solved, with constants of integration being determined by poloidal periodicity constraints. Finally in second order, the solubility condition on $\Psi^{(2)}$ provides a flux-surface averaged equation for $\Psi^{(0)}$ after substituting for $\Psi^{(1)}$. However this also involves flux-surface averages of velocity moments of the magnetic drift term in the gyro-kinetic equations which can be evaluated by repeated integrations by parts in poloidal angle, use of the gyro-kinetic equations up to third order and noting conservation of momentum in ion-ion collisions. These manipulations give rise to terms that can be recognised as (i) the enhanced neoclassical ion inertia which adds to that already present in the vorticity equation due to the usual ion polarisation drift, (ii) ion neoclassical cross-field viscosity (ignored in our present ordering), dominated by the perturbed ion temperature gradient and (iii) a term arising from the parallel gradient of perturbed pressure. This equation provides another second order ODE linking $\Phi^{(0)}$ and $\Psi^{(0)}$ involving $\hat{p}$ and $\hat{p}'$, where $\hat{p} = \sum_j \hat{p}_j = \sum_j \int d^3v\left(m_j v^2/3\right)h_{j0}$. In the notation of Ref. 6,

$$x\left(d^2\left(x\Psi^{(0)}\right)\Big/dx^2\right)+D_I\Psi^{(0)}-\left(x\hat{p}\right)'\left(\left(L+H\right)/p_0'\right)+D_I\left(\omega\hat{p}/n\right)=$$

$$\left(m_i n_i\omega^2/n^2\right)\left(V'/q'\right)^2\left\langle B^2\Big/\left|\nabla\psi\right|^2\right\rangle\left\{\left\langle R\right\rangle^2\left(1-\omega_{*pi}/\omega\right)+\left(1.17f_c I^2\Big/\left\langle B\right\rangle^2\right)\omega_{*pi}\eta_i/\omega\right\}\left(d^2\Phi^{(0)}/dx^2\right) \qquad (5)$$

3. The collisional limit

The general eigenvalue problem obtained by this method has the form of an eighth order system of ODE's. Ignoring ion neoclassical thermal transport and viscosity simplifies this to a fourth order system. However it is interesting to take the collisional limit, $Q \to 0$, since this generalises Refs. 2, 3 by including temperature gradient effects. Then the expressions for $\hat{n}_j$ and $\hat{T}_j$ are simpler and Fourier transforming allows one to reduce the Ohm's law and vorticity equations to a single second order ODE. This differs from Refs.2, 3 by the substitution:

$$\omega-\omega_{*e} \to \omega-\omega_{*e}\left[1+\left(5.9-1.7f_c\right)\eta_e\Big/\left(2.45-0.45f_c\right)\right] \qquad (6)$$

in Ohm's law and the inertia term in eqn. (5). It is worth recalling that Ref. 2 identified a strong reduction in growth rate of the Δ' driven tearing mode due to neoclassical resistivity and ion inertia effects, although the bootstrap current drive in the resonant layer reverses the Glasser stabilisation effect [6] and leads to an unstable tearing parity 'interchange' mode, a linear analogue of finite island neoclassical tearing mode.

4. Discussion and Conclusions

We have derived a set of equations that describe the linear stability of tearing modes in the low collisionality regime appropriate to large tokamaks such as JET or ITER. Although these have the form of fluid-like equations for moments such as plasma density, temperature and current to feed into Maxwell's equations, they contain coefficients that encapsulate kinetic neoclassical effects, While the electron model corresponds to the semi-collisional regime in which parallel transport effects compete with the mode frequency ($\omega \sim k_{\parallel}^2 v_{the}^2 / \nu_e$), the fluid ion treatment can only be justified for low magnetic shear, s. Using the semi-collisional definition of the electron layer width, δ_{sc}, $\omega_{*e} \sim \left(k_\theta \delta_{sc} s / Rq\right)^2 v_{the}^2 / \nu_e$, this implies $s < \left[\sqrt{m_e/m_i}\left(a/R\right)^{3/2}\left(Rq/L_n\right)\left(\nu_{*e}/\rho_{*i}\right)\right]^{1/2}$. Nevertheless this is highly relevant for describing the resistive internal kink mode involved in the sawtooth phenomenon. (For finite shear one would need a finite banana width treatment in the spirit of Ref. 7.) One can identify a number of characteristic lengths: the semi-collisional layer width, $\delta_{sc} \sim \left[\left(\varepsilon^{3/2}\nu_{*e}/s^2\right)\left(Rq/L_n\right)\rho_e r\right]^{1/2}$, the resistive layer width, $\delta_\eta \sim \left(\eta/\omega\right)^{1/2} \sim \left[\left(\varepsilon^{3/2}\nu_{*e}/\beta_e\right)\left(L_n/Rq\right)\rho_e r\right]^{1/2}$, and the ion neoclassical transport length scale $\delta_{\chi_i} \sim \left(\varepsilon^{-3/2}\nu_i \rho_i^2 q^2/\omega\right)^{1/2} \sim \left[\nu_{*i}\left(qL_n/R\right)\rho_i r\right]^{1/2}$ (we will address the effect of neoclassical ion transport and toroidal viscosity in later work). One expects the collisional model to pertain if $\delta_\eta > \delta_{sc}$; in fact the ratio $\left(\delta_\eta/\delta_{sc}\right)^2 \sim \left(s/\beta_e\right)\left(L_T/R\right)$ is likely to be O(1) for a sawtooth with s~ 0.1, so one must indeed consider semi-collisional effects when modelling the sawtooth instability. The implications of the complete set of physical effects we have included remain to be calculated.

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