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THE RADIAL DRIFT INVARIANT IN MIRRORS IN THE PARAXIAL APPROXIMATION

O.Ågren[1],V.E.Moiseenko[1,2]

[1] *Uppsala University, SE-751 21Uppsala, Sweden*

[2] *National Science Center "Kharkov Institute of Physics and Technology"*

61108 Kharkiv, Ukraine

A set of three constants of motion (the energy, the magnetic moment and the parallel action integral) are typically used to describe plasma confined by a magnetic mirror field. Ryutov and Stupakov [1] have used these three invariants to determine criteria for omnigenuity and describe neoclassical transport effects in quadrupolar mirror fields.

We have recently shown that under rather general conditions (valid for mirrors as well as tokamaks) there exists a fourth independent constant of motion [2], i.e. a radial drift invariant. The existence of the radial drift invariant is assured by the requirement that the guiding center of a confined particle cannot have a net radial drift, and thus only oscillatory guiding center radial displacements off an initial magnetic flux surface can appear. To lowest order in the radial displacements off the magnetic surface, the radial drift invariant corresponds to a guiding center motion on the flux surface. In tokamaks, the new invariant provides a simple mean to describe the poloidal current from the Vlasov equation, and a bridge to ideal MHD equilibria has been established by using a local Maxwellian distribution function expressed in terms of the new constant of motion. This same form of the distribution function can also be used to establish a bridge between neoclassical transport theory in tokamaks and the collision free state.

Similar theoretical results for quadrupolar mirror fields are presented. The vacuum field $\mathbf{B}_\mathrm{v} = \nabla\chi$ satisfies $\nabla\chi\big|_{x=y=0} = \tilde{B}(z)\hat{z}$ and $\nabla^2\chi = 0$. In the notation of Ryutov and Stupakov, a quadrupolar expansion of χ to second order in x and y (the long-thin approximation) is

$$\frac{\chi(x,y,z)}{B_0} = \int_0^z R_m dz + \frac{x^2}{2} R_m \frac{X'}{X} + \frac{y^2}{2} R_m \frac{Y'}{Y} + \mathrm{O}(\lambda^4) \qquad (1)$$

where $B_0 = \tilde{B}(0)$, $R_m(z) = \tilde{B}/B_0 = 1/(XY)$ is the mirror ratio along the z axis and primes denotes differentiation with respect to z. We are free to choose and $X(0) = Y(0) = 1$.

The leading order flux line equations are

$$x(z) = x_0 X(z) \quad , \quad y(z) = y_0 Y(z),$$

With a circular intersection of the flux tube with the midplane $z = 0$, the flux surface is determined by $a^2 = x_0^2 + y_0^2 = [x / X(z)]^2 + [y / Y(z)]^2$. If the arc length s along the field lines satisfies $|\nabla s| \to 1$ at $z = 0$ we obtain

$$\nabla s = \hat{\mathbf{B}} + \tilde{a}_0(z) \frac{x_0}{c} \nabla x_0 + \tilde{b}_0(z) \frac{y_0}{c} \nabla y_0$$

where

$$\tilde{a}_0(s) = \int_0^s ds\, c \frac{d^2 X(s)}{ds^2} X(s) \quad , \quad \tilde{b}_0(s) = \int_0^s ds\, c \frac{d^2 Y(s)}{ds^2} Y(s)$$

A quadrupolar expansion of guiding center potential $U_{gc} = q\phi + \mu B$ reads in the (x_0, y_0, s) coordinates

$$U_{gc} = U_0(s) + \frac{x_0^2}{2c^2} \tilde{U}_1(s) + \frac{y_0^2}{2c^2} \tilde{U}_2(s)$$

The energy conservation is

$$\varepsilon = U_{gc}[\overline{x}_0(\lambda t), \overline{y}_0(\lambda t), \lambda \overline{s}(t)] + \frac{m}{2} \overline{\mathrm{v}}_{\|}^2 = const,$$

and the time evolution of the guiding center coordinates $(\overline{x}_0, \overline{y}_0, \overline{s})$ becomes

$$\frac{d\overline{s}}{dt} \equiv \overline{\mathrm{v}}_{\|} + [\tilde{a}_0(s) \frac{\overline{x}_0}{c} \frac{d\overline{x}_0}{dt} + \tilde{b}_0(s) \frac{\overline{y}_0}{c} \frac{d\overline{y}_0}{dt}]$$

$$\frac{d\overline{x}_0}{dt} = -\omega_2(s) \overline{y}_0 \quad , \quad \frac{d\overline{y}_0}{dt} = \omega_1(s) \overline{x}_0$$

Introducing cylindrical Clebsch coordinates by $\overline{x}_0 = \overline{r}_0 \sin \overline{\theta}_0$ and $\overline{y}_0 = \overline{r}_0 \cos \overline{\theta}_0$, the last two equations can be written

$$\frac{d\overline{r}_0}{dt} = (\omega_1 - \omega_2) \overline{r}_0 \sin \overline{\theta}_0 \cos \overline{\theta}_0 \quad , \quad \frac{d\overline{\theta}_0}{dt} = \omega_1 \cos^2 \overline{\theta}_0 + \omega_2 \sin^2 \overline{\theta}_0 .$$

When the electric field gradients are neglected we have

$$\omega_1(s) = \frac{\mu}{q} (\frac{3}{XY} - \frac{2\varepsilon}{\mu B_0}) XX'' \quad , \quad \omega_2(s) = \frac{\mu}{q} (\frac{3}{XY} - \frac{2\varepsilon}{\mu B_0}) YY''$$

Axisymmetric fields obey the symmetry $\omega_1(s)=\omega_2(s)$, and in such fields the guiding center radial coordinate $\bar{r}_0$ is a constant of motion. Another field in which $\bar{r}_0$ is constant is the straight field line mirror, where $X(s)\to 1+s/c$, $Y(s)\to 1-s/c$ and $\omega_1(s)=\omega_2(s)=0$, implying that both $\bar{x}_0$ and $\bar{y}_0$ are constant and the perpendicular drift vanish completely in that particular field.

For quadrupolar fields with $\omega_1(s)\omega_2(s)\neq 0$, we arrive at harmonic oscillator equations;

$$[\frac{d^2}{d\tau_2^2}+\frac{\omega_2^2(0)\omega_1(s)}{\omega_2(s)}]\bar{x}_0 = 0 \quad , \quad [\frac{d^2}{d\tau_1^2}+\frac{\omega_1^2(0)\omega_2(s)}{\omega_1(s)}]\bar{y}_0 = 0$$

where $\tau_1(t) = \int^t dt\, \omega_1[s(t)]/\omega_1(0)$ and $\tau_2(t) = \int^t dt\, \omega_2[s(t)]/\omega_2(0)$. In the absence of electric field terms, we have $\omega_1(s)/\omega_2(s)=XX''/YY''$. If $\omega_1(s)/\omega_2(s)$ and its reciprocal $\omega_2(s)/\omega_1(s)$ are slowly varying, which is satisfied if the drift frequency around the magnetic axis is large compared to the longitudinal bounce frequency, WKB solution becomes

$$\bar{x}_0 \approx \sqrt{\Gamma(s)}\, I_r(\mathbf{x},\mathbf{v})\cos\bar{\theta}_0 \quad , \quad \bar{y}_0 \approx \frac{I_r(\mathbf{x},\mathbf{v})\sin\bar{\theta}_0}{\sqrt{\Gamma(s)}}$$

where $\bar{\theta}_0=\bar{\theta}_0(0)+\int_0^t dt\sqrt{\omega_1[s(t)]\omega_2[s(t)]}$, $I_r(\mathbf{x},\mathbf{v})$ and $\bar{\theta}_0(0)$ are constant and

$$\Gamma(s) = \Gamma(s,\varepsilon,\mu) = \sqrt{\frac{\omega_2(s)/\omega_2(0)}{\omega_1(s)/\omega_1(0)}}$$

This describes a guiding center motion on a drift surface with elliptic cross sections,

$$I_r(\mathbf{x},\mathbf{v}) = \sqrt{\bar{x}_0^2/\Gamma^2(s) + \bar{y}_0^2\Gamma^2(s)} = constant$$

Typically, the ellipticities $\varepsilon_{ell,drift}(z)$ and $\varepsilon_{ell}(z)$ of the drift and magnetic surfaces differ. The condition $\varepsilon_{ell,drift}(z)=\varepsilon_{ell}(z)$ has been used by Panov [3] and Skovoroda [4] as a criterion for omnigenuity in the paraxial approximation.

Constant values of the radial coordinate determine flux surfaces, and the guiding center satisfies

$$\bar{r}_0 = \bar{r}_{0,mean}(s)[\ 1 + \frac{\Gamma^2(s)-1}{\Gamma^2(s)+1}\frac{\cos 2\bar{\theta}_0}{2}\ ]$$

where $\overline{r}_{0,mean}(s) \equiv \overline{r}_{0,mean}(s,\varepsilon,\mu,I_r) = I_r(\mathbf{x},\mathbf{v})\sqrt{(\Gamma^2+1)/2\Gamma}$ may differ from the constant I_r. This means that the mean drift surface deviate from the flux surface, although the two surfaces intersect at the midplane $s \approx z = 0$. The difference

$$\Delta\overline{r}_{0,mean}(s,\varepsilon,\mu,I_r) \equiv \overline{r}_{0,mean}(s) - I_r \equiv [\sqrt{\frac{\Gamma^2(s)+1}{2\Gamma(s)}} - 1] I_r$$

may add a contribution to the neoclassical transport, even though the banana widths are zero.

The radial invariant can be used to construct equilibria with distribution functions of the form $F(\varepsilon,\mu,I_r)$. The pressure perturbed magnetic field is given by

$$\mathbf{B} = (1-\frac{\beta}{2})(\mathbf{B}_v + \nabla\phi_{\beta,pl}) \quad , \quad \phi_{\beta,pl}(\mathbf{x}) = -\frac{1}{8\pi}\int \frac{\nabla'\beta(\mathbf{x}')\cdot\mathbf{B}_v(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$$

where $\beta = 2\mu_0 P_\perp / B_v^2$. As seen, no parallel current or finite banana widths are obtained to leading order.

Conclusions

An expression for the radial invariant in mirrors have been derived. An orbit effect associated with a difference between the ellipticities of the flux and mean drift surfaces can enhance the neoclassical transport. The explicit solution for the magnetic field shows that there is no parallel current or finite banana with to leading orders with distribution functions of the form $F(\varepsilon,\mu,I_r)$.

References

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