The equilibrium equation in skew configurations

E.A. Evangelidis[1], G.J.J. Botha[2]

[1] *Demokritos University of Thrace, Euratom-Hellenic Association, Xanthi, Greece*

[2] *University of Leeds, Leeds, United Kingdom*

Introduction

The Shafranov shift $\triangle$ of the magnetic axis in toroidal devices is one of the fundamental parameters in the description of plasma equilibrium. Wesson [4] provides an exhaustive treatment of the case when $\triangle$ is constant. In the case where this shift is a function of the minor radius, a preliminary analysis was presented by Sandquist [2, 3]. This latter case is treated at length in a communication to be submitted in the future, with only the salient points presented here.

Non-orthogonal Shafranov coordinate system

In Shafranov coordinates a point on a torus is determined by the position vector

$$\mathbf{r} = R\cos\phi\,\hat{\mathbf{i}} + R\sin\phi\,\hat{\mathbf{j}} + \rho\sin\theta\,\hat{\mathbf{k}}, \quad \text{with} \quad R = R_0 + \rho\cos\theta - \triangle(\rho), \tag{1}$$

where the coordinates (ρ, ϕ, θ) have their standard meaning. Using subscripts to denote differentiation with respect to the corresponding variable, the derivatives

$$\mathbf{r}_\rho = \Big[\big(\cos\theta - \triangle'\big)\cos\phi,\ \big(\cos\theta - \triangle'\big)\sin\phi,\ \sin\theta \Big], \tag{2}$$

$$\mathbf{r}_\phi = R\,[-\sin\phi,\ \cos\phi,\ 0], \tag{3}$$

$$\mathbf{r}_\theta = \rho\,[-\sin\theta\cos\phi,\ -\sin\theta\sin\phi,\ \cos\theta], \tag{4}$$

follow. The Jacobian of the system is defined as

$$J = \mathbf{r}_\rho \cdot (\mathbf{r}_\phi \wedge \mathbf{r}_\theta) = \rho R (1 - \triangle'\cos\theta). \tag{5}$$

Using these results, one can construct three non-orthogonal unit directions $(\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_\theta)$, along which a vector field is to be analyzed. Their vector properties are

$$\mathbf{e}_\rho \cdot \mathbf{e}_\phi = 0, \qquad \mathbf{e}_\phi \cdot \mathbf{e}_\theta = 0, \qquad \mathbf{e}_\theta \cdot \mathbf{e}_\rho = -\triangle'\sin\theta, \tag{6}$$

$$\mathbf{e}_\rho \wedge \mathbf{e}_\phi = \mathbf{e}_\theta + \triangle'\sin\theta\,\mathbf{e}_\rho, \qquad \mathbf{e}_\phi \wedge \mathbf{e}_\theta = \mathbf{e}_\rho + \triangle'\sin\theta\,\mathbf{e}_\theta, \qquad \mathbf{e}_\theta \wedge \mathbf{e}_\rho = \mathbf{e}_\phi. \tag{7}$$

From the expressions for the vectorial operators in Brand [1], it is easily found that

$$\nabla = \frac{\mathbf{e}_\rho}{1 - \triangle'\cos\theta}\frac{\partial}{\partial\rho} + \frac{\mathbf{e}_\phi}{R}\frac{\partial}{\partial\phi} + \frac{\mathbf{e}_\theta}{\rho}\frac{\partial}{\partial\theta}, \tag{8}$$

$$\nabla \cdot \mathbf{f} = \frac{1}{J}\left\{ \frac{\partial}{\partial\rho}\Big[\rho R\,\mathbf{e}_\rho \cdot \mathbf{f}\Big] + \frac{\partial}{\partial\phi}\Big[\rho(1 - \triangle'\cos\theta)\,\mathbf{e}_\phi \cdot \mathbf{f}\Big] + \frac{\partial}{\partial\theta}\Big[R(1 - \triangle'\cos\theta)\,\mathbf{e}_\theta \cdot \mathbf{f}\Big] \right\}, \tag{9}$$

$$\nabla \wedge \mathbf{f} = \frac{1}{J}\left\{ \frac{\partial}{\partial\rho}\Big[\rho R\,\mathbf{e}_\rho \wedge \mathbf{f}\Big] + \frac{\partial}{\partial\phi}\Big[\rho(1 - \triangle'\cos\theta)\,\mathbf{e}_\phi \wedge \mathbf{f}\Big] + \frac{\partial}{\partial\theta}\Big[R(1 - \triangle'\cos\theta)\,\mathbf{e}_\theta \wedge \mathbf{f}\Big] \right\}. \tag{10}$$

At this point we have to clarify the meaning of the projection of a vector in a non-orthogonal system of coordinates. A vector $\underline{\mathbf{AB}}$ in an orthogonal reference system can be represented as $\underline{\mathbf{AB}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$, where x and y are the measured lengths along $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. Indeed, the projection of $\mathbf{AB}$ onto the orthogonal axes gives $\hat{\mathbf{i}} \cdot \mathbf{AB} = x$ and $\hat{\mathbf{j}} \cdot \mathbf{AB} = y$, since $\mathbf{i} \cdot \mathbf{j} = 0$. Clearly in the Shafranov system the vectorial representation of a vector field should be different. We shall look for such an expression under the requirement that the projection of $\mathbf{f}$ onto the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, with $\mathbf{e}_1 \cdot \mathbf{e}_2 = \cos\theta_0$, would reproduce the measured components f^1 and f^2 in the $(\mathbf{e}_1, \mathbf{e}_2)$-plane. After some elementary operations the result

$$\mathbf{f} = \frac{1}{\sin^2\theta}\left[\left(f^1 - f^2\cos\theta_0\right)\mathbf{e}_1 + \left(f^2 - f^1\cos\theta_0\right)\mathbf{e}_2\right] \tag{11}$$

follows. Since $\cos\theta_0 = -\triangle'\sin\theta$ from (6), the sought for expression becomes

$$\mathbf{f} = \left(f^\rho + f^\theta\triangle'\sin\theta\right)\mathbf{e}_\rho + f^\phi\mathbf{e}_\phi + \left(f^\theta + f^\rho\triangle'\sin\theta\right)\mathbf{e}_\theta, \tag{12}$$

where we have restored the third dimension, went over to the Shafranov coordinates, and omitted second order terms in $\triangle'$. Since the above representation was derived on the requirement that its projection should reproduce the measured, physical components of the field, we can write down immediately the expression for the divergence:

$$\nabla\cdot\mathbf{f} = \frac{1}{J}\left\{\frac{\partial}{\partial\rho}\left[\rho R f^\rho\right] + \frac{\partial}{\partial\phi}\left[\rho(1-\triangle'\cos\theta)f^\phi\right] + \frac{\partial}{\partial\theta}\left[R(1-\triangle'\cos\theta)f^\theta\right]\right\}. \tag{13}$$

For the calculations involved in the derivation of the curl operator, one uses expressions (7), as well as the derivatives of the unit vectors as the point travels in the space of the torus, or on the isoflux surfaces, to eventually arrive at the result

$$\begin{aligned}\nabla\wedge\mathbf{f} =\; & \left[\frac{1}{R}\frac{\partial f^\theta}{\partial\phi} - \frac{1}{\rho R}\frac{\partial}{\partial\theta}\left(Rf^\phi\right) + \frac{\triangle'\sin\theta}{R}\frac{\partial}{\partial\rho}\left(Rf^\phi\right)\right]\mathbf{e}_\rho + \\ & \left\{\frac{1}{\rho}\frac{\partial}{\partial\theta}\left(f^\rho + f^\theta\triangle'\sin\theta\right) - \frac{\partial}{\partial\rho}(f^\theta + f^\rho\triangle'\sin\theta) - \frac{1}{\rho}\left[f^\theta + \triangle'\cos\theta\frac{\partial}{\partial\rho}\left(\rho f^\theta\right)\right]\right\}\mathbf{e}_\phi \\ & + \left\{\frac{1+\triangle'\cos\theta}{R}\left[\frac{\partial}{\partial\rho}\left(Rf^\phi\right) - \frac{\partial f^\rho}{\partial\phi}\right] + \frac{\triangle'}{R}\left[\cos\theta\frac{\partial f^\rho}{\partial\phi} - \frac{\sin\theta}{\rho}\frac{\partial}{\partial\theta}\left(Rf^\phi\right)\right]\right\}\mathbf{e}_\theta.\end{aligned} \tag{14}$$

Plasma equilibrium in Shafranov coordinates

For an axisymmetric plasma in equilibrium, the equations to be analyzed are Ampère's equation and the relation between the magnetic and plasma pressure, complemented by the divergence free magnetic and current fields:

$$\mathbf{j}\wedge\mathbf{B} = c\nabla p, \qquad \nabla\wedge\mathbf{B} = \frac{4\pi}{c}\mathbf{j}, \tag{15}$$

$$\nabla\cdot\mathbf{B} = 0, \qquad \nabla\cdot\mathbf{j} = 0. \tag{16}$$

From the pressure balance equation the relation

$$-\frac{1}{\rho}\frac{\partial p}{\partial \theta} = \frac{1}{1-\triangle'\cos\theta}\frac{B^{\rho}}{B^{\theta}}\frac{\partial p}{\partial \rho} \tag{17}$$

is deduced, showing that the radial gradient of pressure has to be proportional to the gradient of pressure along the minor section. Equations (16) are simultaneously satisfied by the two scalar potential functions ψ and f, such that

$$B^{\rho} = -\frac{1}{\rho R}\frac{\partial \psi}{\partial \theta}\,, \qquad B^{\theta} = \frac{1}{R(1-\triangle'\cos\theta)}\frac{\partial \psi}{\partial \rho}\,, \tag{18}$$

$$j^{\rho} = -\frac{1}{\rho R}\frac{\partial f}{\partial \theta}\,, \qquad j^{\theta} = \frac{1}{R(1-\triangle'\cos\theta)}\frac{\partial f}{\partial \rho}\,. \tag{19}$$

Furthermore, from the radial and poloidal components of Ampère's equation the relation

$$B^{\phi} = \frac{4\pi}{cR} f \tag{20}$$

was found to exist between the current density function and the azimuthal component of the magnetic field. Substituting the above expressions in the azimuthal component of Ampère's equation, it follows that

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{1}{R}\left[\frac{\sin\theta}{\rho}\frac{\partial \psi}{\partial \theta} - \cos\theta\frac{\partial \psi}{\partial \rho}\right] = -4\pi R^2\frac{\partial p}{\partial \psi} - \left(\frac{4\pi}{c}\right)^2 f\frac{\partial f}{\partial \psi}\,. \tag{21}$$

This is the sought for equation describing the state of equilibrium in the Shafranov equation. It coincides formally with the expression for the orthogonal configuration, referred to earlier, where the Shafranov shift $\triangle$ is constant and the radial and poloidal directions are orthogonal.

References

[1] L. Brand, Vector and Tensor Analysis, John Wiley & Sons, (1947)

[2] P. Sandquist, Alfvén Eigenmodes, Sawtooth Oscillations and Fast Ion Confinement in Alpha Simulation JET Experiments. A thesis for the degree of MSc in Engineering Physics, Chalmers University of Technology, Göteborg, Sweden (2004)

[3] S. Sharapov, private communication (2006)

[4] J. Wesson, Tokamaks, Clarendon Press, Oxford, p.118 (2004)