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Limits and potentialities of Fokker-Planck Equation in describing anomalous transport

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Fokker-Planck Equation (FPE) is a basic model for the description of transport processes. In one dimension it reads

$$\frac{\partial n(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left(V(x)n\right) + \frac{\partial^2}{\partial x^2}\left(D(x)n\right) \qquad (1)$$

where n is the density for a generic scalar quantity, V the dynamic friction, and D the diffusion coefficient. However FPE misses some features of the true transport. This was proved, e.g., for the transport of tracer particles suddenly released in pressure-gradient-driven turbulence, which exhibits strongly non gaussian features [1]. On the other hand, some doubts were casted (see [2] and references therein) about the ability of Eq. (1) to describe some anomalous features of transport in tokamaks, including non-diffusive propagation of disturbances [3], anomalous scaling of confinement time with device size [4], and uphill transport [5]. These facts set therefore the issue: when is FPE relevant for plasma transport, when is it not? This paper gives the main results useful for experimentalists and numericists of a recent work [6] dealing with this issue.

We first show that FPE has the ability to display the "anomalous" features reported in refs. [3-5]. The relevant dimensionless parameter is now $C = L_0V_0/D_0$, with L_0 , V_0 , D_0 , respectively the typical values for the device size, the friction, and the diffusivity. When $C \gg 1$ the dynamics of the system is driven by its convective part, hence more-than-diffusive (ballistic) spreading of a perturbation is automatically accounted for. The same relative balance governs the scaling of the confinement time τ_{conf} : in a diffusively-driven system ($C < 1$), $\tau_{conf} \propto L_0^2$, whereas in the convectively-driven one ($C > 1$), $\tau_{conf} \propto L_0^1$. The space dependence of the transport coefficients and their connection to gradients in the plasma can bring the scaling $\tau_{conf} \propto L_0^{\alpha}$, $\alpha \le 2$ suggested by experiments [4]. Finally, let us write (1) in stationary conditions and in absence of sources as

$$\frac{\partial n/\partial x}{n} = \frac{V - \partial D/\partial x}{D} \qquad (2)$$

and let $0 < x < 1$ be the normalized radius. As an instance of "uphill transport" we consider the central peaking of the density. It is provided by several choices of the couple (V, D): e.g., $V < 0$ and $|V| > |dD/dx|$ for whatever sign of dD/dx, or $V = 0$ and $dD/dx > 0$. This is a caveat for data analysis: a broad family of (V,D) profiles can model the same experimental data.

We now discuss the possible link between V and D. Landau introduced the constraint, hereafter referred to as Landau Constraint (LC) [7]

$$V(x) = \frac{dD(x)}{dx} \tag{3}$$

reducing (1) to its Fickian version $\partial n(x,t)/\partial t = (\partial/\partial x)\big(D(x)\partial n(x,t)/\partial x\big)$. This constraint holds rigorously for one degree of freedom (dof) Hamiltonian dynamics. This was shown in particular instances [7], and more recently in a general way [6]. LC rules out uphill transport. However, as soon as the number of dof increases, LC breaks down. This is bound to occur in a plasma, since reducing particle motion to 1 dof requires strong approximations.

FPE may be deduced as a small-step-length limit of the Chapman-Kolmogorov Equation:

$$\frac{\partial n(x,t)}{\partial t} = -\frac{n(x,t)}{\tau(x)} + \int P(x,x')\frac{n(x',t)}{\tau(x')}dx' + S(x) \tag{4}$$

Neglecting any spatial dependence in the waiting time τ ($\tau = 1$), Eq. (4) depends upon three spatial scales: L_0, the container size; L_P, the typical width for the displacement $x - x'$, driven by P; L_S, the width of the source S. Equation (1) is recovered under the limit $L_P << L_S$, L_0 . If $L_S < L_P < L_0$, solving Eq. (4) in the Fourier space and under steady-state conditions shows that, for $k \sim 1/L_S$, $n(k) \sim S(k)$: the density matches the profile of the source. Being the source highly localized, this means that strong gradients must be pumped in $n(x)$, which cannot be justified on the basis of FPE transport coefficients. Such a scenario could be of relevance in the case of edge fuelling in the presence of radially elongated turbulent structures (streamers). This is a further caveat for data analysis.

We now show how FPE can be justified for particle dynamics. Transport across the magnetic field induced by electrostatic turbulence in the guiding center approximation is described by

$$\frac{d\mathbf{X}}{dt} = -\nabla\Phi(\mathbf{X},t)\times\mathbf{x}_3 \equiv \mathbf{v} \tag{5}$$

with $\mathbf{X} = (x_1,x_2)$ the particle position perpendicular to the magnetic field and Φ the appropriately normalized electrostatic potential. Φ is a statistically stationary, spatially homogeneous, isotropic zero-mean-value stochastic potential with typical amplitude Φ_0, correlation time τ_c and correlation length λ_c. An extended study of this system may be found in [8] which shows its diffusive behavior. Two different diffusion scenarios can be found, depending on the value of the Kubo number $K = \Phi_0\tau_c/\lambda_c^2$. If the potential is static (τ_c infinite), particles are trapped into potential wells and hills, or can make long flights along *roads* crossing the whole chaotic domain. The various domains are separated by separatrices joining nearby hyperbolic points. As soon as τ_c becomes finite, but large ($K >> 1$), the potential topography slowly changes, and the dynamics evolves quasi-adiabatically. Since phase space area inside the instantaneous closed orbits must be adiabatically preserved, and since the area of the various domains defined by the separatrix array fluctuates a lot, orbits must cross the instantaneous separatrices, and jump this way from one domain to the next one (Fig. 1a). These crossings produce a random walk with step with $D \approx \lambda_c^2/\tau_c$ in the absence of roads [8]. As a result, for $K >> 1$ diffusion is justified by locality of trapping in phase-space. When $K << 1$, the particles typically run only along a small arc of length $\Phi_0\tau_c/\lambda_c$ of the trapped orbits of the instantaneous potential during a correlation time. During the next correlation time they perform a similar motion in a potential completely uncorrelated with the previous one (Fig. 1b). These uncorrelated random steps still yield a 2D Brownian motion with diffusion coefficient $D \approx \Phi_0^2\tau_c/\lambda_c^2$. These two limit cases in K show diffusion is a quite general behavior of particle transport, even whenever structures are visible in one realization of the electrostatic potential, or whenever non gaussian behaviour is obtained for a more limited statistics, as in Ref. [1] which used dynamics of type (5), but with an anisotropic potential. It is important to notice that, *depending on the statistics of interest, the same dynamical system may be found diffusive or dominated by its Lévy flights, and thus non FPE-relevant*.

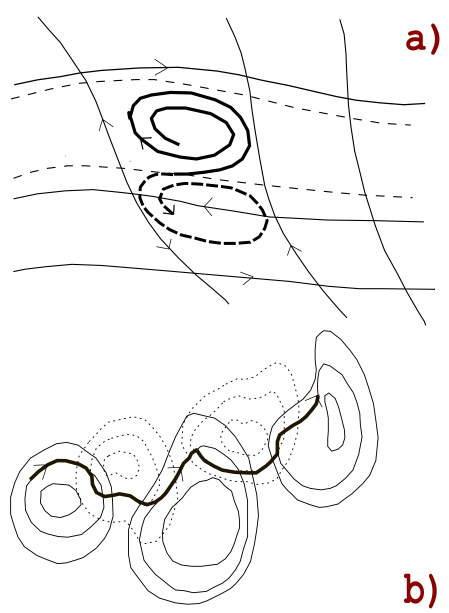

Figure 1. (a) $K >> 1$: as the net of intersecting separatrices evolves in time (solid to dashed curves), the lower domain enlarges at the expenses of the upper one. A trajectory may go from the upper to the lower one. (b) $K << 1$: jumps among small trapping arcs in a quickly evolving potential.

We now consider how a dynamic friction may occur whan a spatial inhomogeneity is introduced in model (5). To this end its r.h.s. is multiplied by a growing function of x_1 [9]. This may be meant as a modeling of the increase of the magnetic field toward the main axis of a fusion machine, but makes the dynamics non Hamiltonian. On top of the previous diffusive behavior (which becomes anisotropic), the new inhomogeneity brings a radial drift velocity *V* along x_1 due to the chaotic motion, which corresponds to the dynamic friction of FPE. The sign of *V* depends on *K*. If $K << 1$, since the velocity increases toward larger x_1's, the displacement during a correlation time is larger toward the exterior than toward the interior, which brings an outgoing drift. If $K >> 1$, the trapped particles are slower in the inner part of their orbit, which increases their probability to be there with respect to that to be in the outer part: this brings an ingoing drift. Since *D* now grows with x_1, $V = dD/dx_1$ is impossible for $K>>1$. This shows a high flexibility in the definition of *V*.

As a result, we claim FPE can be justified for generic particle transport provided that there is enough randomness in the Hamiltonian describing the dynamics. Since fusion experiments generally average over many plasma realizations, and global confinement scaling laws even more so, FPE is a highly relevant tool for this field. This is a caveat for numerical simulations where the statistics may be scarcer than in experiments: numerical non gaussian pdf's may not be typical of a corresponding experiment, but a mere consequence of a lack of averaging. Phase locking may prevent a broad averaging and invalidate FPE. This occurs in [1] and for particle dynamics in the multiple helicity state of the Reversed Field Pinch [10].

It should be noted that in a magnetized toroidal plasma, density *n* describing particle transport is the true particle density divided by a growing function $\tau(x)$ of the local magnetic field (see [11] and references therein). This brings a slanting of the density profile toward the outer part of the torus that has nothing to do with a turbulent transport phenomenon, in contrast with that in [9], and which corresponds exactly to a waiting time $\tau(x)$ in Eq. (4). This is one more caveat for data analysis.

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