*34th EPS Conference on Plasma Phys. Warsaw, 2 - 6 July 2007 ECA Vol.**31F**, P-2.076 (2007)*

A first order Godunov scheme for special relativistic fluid dynamics

Danilo Durante, Giorgio Riccardi

Dept. of Aerospace and Mech. Eng., Second University of Naples, Italy

Introduction

The special relativistic equations of rest mass, momentum and energy conservation can be written in conservative form as

$$\begin{cases} \partial_t\, D + \partial_{x_k}\,(Du_k) & = & 0 \\ \partial_t\, S_j + \partial_{x_k}\,(S_j u_k + p\delta_{jk}) & = & 0 \\ \partial_t\, \tau + \partial_{x_k}\,(S_k - Du_k) & = & 0 \end{cases} \tag{1}$$

where: $D = \rho_0 w$, $S_j = Dw\mu u_j$ and $\tau = \rho_0 \mu w^2 - p - \rho_0 w$. The variables μ (total specific entalpy) and w (Lorentz factor) are $\mu = 1 + \gamma p/[\rho_0(\gamma - 1)]$ (γ is the specific heats ratio) and $w = 1/\sqrt{1 - u_k u_k}$. As it is evident by the expression of the total specific entalpy, the state equation of an ideal gas have been used.

In order to perform a numerical integration of (1), in the last decades Godunov type schemes have been widely adopted, because of their formal simplicity and great numerical stability. In the framework of a Godunov scheme, fluxes at the interface between numerical cells are evaluated by solving Riemann problems. The Riemann problem concerns the relaxation of a field which is discontinuous across a plane at the initial time $t = 0$. Three kind of waves arise, two of which can be rarefaction or shock waves and a third one which is a contact discontinuity. For the sake of clearness suppose that this waves travel along x_1 direction (*i.e.* the plane is the (x_2, x_3) one), so that the solution of a Riemann problem can be analyzed in the meaningfull (u_1, p)-plane. Once the state *ahead* the wave (say state 1) is fixed, you can find all the possible states *behind* (say 2). The wave pattern which follows will be indicated as belonging to the first family; in this way shock waves or rarefactions of the first family will be referred respectively to SCN or RCN curves. Similarly we can consider the same state (say 4) as the one *behind* the wave, in order to find all the possible states *ahead* (say 3). In this last case it will be considered NCS or NCR curves of the second family.

The foundamental task of a Riemann solver is to find the intermediate pressure and velocity (state 2 and 3) in a Riemann problem. The knowledge of the wave pattern allows to strongly improve the performances of the solver itself. This issue has been deeply investigated in the paper of Rezzolla and Zanotti [2], where analytical bounds for all the possible wave patterns

have been deduced, by considering the relativistic relative velocity v_{14} of the state 1 with respect to the one 4 (or viceversa). It is proved that v_{14} is a monotonic increasing function of the pressure p_2 (or p_3) in the intermediate states 2 and 3. This analysis allows us to define *a priori* the kind of transformation; however it holds for vanishing tangent velocities ($u_2 = u_3 = 0$) only.

In the framework of the two shock approximation of Colella, the paper of Balsara [3] analyzes the use of two different numerical procedures: secant and Newton methods. By analyzing a lot of numerical tests, Balsara concluded that the Newton method is the best choice to obtain high accuracy, but he never talks about an initialization procedure. On the contrary, our numerical work shows that such procedure is needed, in order to avoid failures during time integration.

Relativistic Riemann solver

The Riemann problem consist in finding the pressure and the velocity into the intermediate state. It has been choosed to find them iteratively, in order to obtain an high precision Riemann solver, so a Newton algoritm has been built. The principal drawback of this code is the necessity of initialize it closer enough to the final solution. Some ad-hoc initialization techniques have been so developed. With a first technique the pressure is discretized with a step Δp. In correspondence with a value $p_k = k\,\Delta p$ of the pressure, the velocities for the first u_k^I and second u_k^{II} families of waves are evaluated. For each pressure interval $[p_{i-1}, p_i)$ the straight lines passing on the points (u_{i-1}^I, p_{i-1}), (u_i^I, p_i) for the first family and on the relative points for the second one are considered and their intersection is evaluated. If the inequality $p_{i-1} \leq p_{int} < p_i$ is verified, the pressure p_{int} is selected as initial data for a Newton method. As second technique, starting by the lower or the upper pressure p_i between the states 1 and 4 (depending by the particular transformation), it can be traced one tangent to the first family curve (relative to the state 1) and another one to the second family curve (relative to the state 4). Once the intersection point of the two lines have been found, the middle pressure $(p_i + p_{int})/2$ is used as new starting point. In this manner the Newton alghoritm can be initialized with a pressure as closer to the analytical one as needed. The foundmental difference between the two techinques lays on a great roboustness of the first, versus a rather significant speed of the second.

First order Godunov scheme

The exact Riemann solver can be efficiently implemented to integrate the Euler equation for an ideal fluid. If you consider a one-dimensional problem along x_1 direction, the spatial domain can be subdivided in n cells of Δx length. By writing the Euler equation in integral form in the $j-$th cell and integrating this equation respect to the time, the following numerical scheme

follows

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left[F(U_{j+1/2}) - F(U_{j-1/2})\right]$$

named *Godunov method,* where $U \equiv (D, S_1, \tau)$ are the volume averaged integration variables and $F \equiv (Du, S_1u_1 + p, S_1 - Du_1)$ is the flux vector. Obviuosly this numerical scheme could be extended to two and three dimensional problems, by using the multidimensional extension of the riemann problem, as found by Martì and Müller [1].

One of the great problems raising in Godunov relativistic numerical scheme is the difficulty to recover the primitive variables u_1, p, ρ_0 by the integration ones D, S_1, τ. Two analytical recoverig procedures have been built. The first one works in velocity and starts with the position $u_1 = \sin\theta$, which is useful to avoid the presence of the square root in the Lorentz factor. With this assumption, the primitive variables can be expressed as functions of θ: $\rho_0 = D\cos\theta$, $p = S_1/\sin\theta - \tau'$ where $\tau' = \tau + D$. In term of $y = \tan\theta/2 \neq 0$, the fourth degree algebraic equation in y is obtained ($\alpha = (\gamma-1)/\gamma$):

$$S_1(1-\alpha)\,y^4 - 2(\tau' + \alpha D)\,y^3 + 2S_1(\alpha+1)\,y^2 - 2(\tau' - \alpha D)\,y - S_1(1-\alpha) = 0\,.$$

Another method works in energy, starting from the definitions of S_1 and τ', by which follows: $u_1 = S_1/(\tau' + p)$. The fourth degree algebraic equation in $\eta = \tau' + p$ is obtained ($\alpha' = 1/\alpha$):

$$(\alpha'-1)^2\eta^4 - 2\alpha'(\alpha'-1)\tau'\eta^3 + [\alpha'^2\tau'^2 + 2(\alpha'-1)S_1^2 - D^2]\eta^2 - 2\alpha'\tau'S_1^2\eta + S_1^2(D^2 + S_1^2) = 0\,.$$

This equation is analytically solved, after a normalization of its coefficients through a division by $\max\{D, |S_1|, \tau'\}$. It has been proved via the Sturm alghoritm that these equations have only one real root physically admissible.

Relativistic mixing layer

A mixing layer arises when the velocity field of two parallel streams, which stationary flow one over the other, is perturbed. The basic velocity field is obtained via an hyperbolic tangent as follows:

$$u_1 = 1/2\,\tanh(20\,x_2/L_{x_2})\,, \quad u_2 = 0\,.$$

On the above basic flow a pertubation field is added. Following the work of Lumpp [4], a solenoidal deterministic pertubation u' is superimposed on the basic field:

$$u_1' = 20\frac{\varepsilon}{\pi}\frac{x_2\lambda}{L_{x_1}^2}\cdot\sin(2\pi\,x_1/\lambda)\cdot\exp(-10\,x_2^2/L_{x_2}^2)\,,\;\; u_2' = \frac{\varepsilon}{2}\cdot\cos(2\pi\,x_1/\lambda)\cdot\exp(-10\,x_2^2/L_{x_2}^2)\,,$$

where ε is the amplitude and λ is the wavelength of the pertubation. The flow is periodic in x_1 direction, while slip conditions (vanishing normal fluxes) have been set on upper ($x_2 = L_{x_2}/2$) and lower ($x_2 = -L_{x_2}/2$) boundaries.