Plasma Ring Structures in Astrophysics: Relevant Theoretical Issues*

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I. Introduction

A basic theory of differentially rotating plasma structures in the prevalent gravitational field of a central object whose mass is M_* has indicated that a "crystal" configuration [1] of the magnetic field produced by internal currents can form around it. Since these structures are substantially more complex than the gaseous accretion disks that are considered usually [2] there are several important issues that remain to be dealt with. Among these we discuss: i) The hierarchy of equations describing the sequence of plasma rings [3] associated with the sequence of counter-streaming current pairs that characterize the crystal magnetic configuration. ii) The poloidal currents that are connected with the presence of a local angular momentum transport process. iii) The effect of local plasma poloidal velocities. iv) The axisymmetric configurations that can be compatible with a mean radial flow velocity as in the case where an accretion process is present.

In this context we may suggest that the innermost double-ring element of the configuration we have considered splits vertically and ejects periodically two rings in the opposite direction. Thus jets could consist of a sequence of macroscopically stable "smoke-rings" propagating along the axis of symmetry in opposite directions.

II. Hierarchy of Equations

The relevant magnetic configuration is represented by $\mathbf{B} \simeq \left[\nabla\psi\times\mathbf{e}_\phi + I(\psi,z)\mathbf{e}_\phi\right]/R_0$ considering a thin structure around a radius $R = R_0$, from the axis of symmetry. We discuss at first the case where only a toroidal internal current is produced and the equilibrium equations are

$$-\frac{\partial}{\partial z}\left(p+\frac{B_R^2}{8\pi}\right)+\frac{1}{4\pi}B_R\frac{\partial}{\partial R}B_z-\rho\Omega_k^2 z=0 \qquad (1)$$

$$2\Omega_k R_0\delta\Omega\rho-\frac{\partial}{\partial R}\left(p+\frac{B_z^2}{8\pi}\right)+\frac{1}{4\pi}B_z\frac{\partial}{\partial z}B_R=0 \qquad (2)$$

where $\Omega_k^2 \equiv GM_*/R^3$ indicates the Keplerian frequency, $\Omega \simeq \Omega_k + \delta\Omega(\hat{\psi})$ is the total rotation frequency, $\hat{\psi}$ is the magnetic surface function associated with the internal currents. We consider $\hat{\psi}$ to be of the form $\hat{\psi} = \tilde{\psi}_0 N(R_*)F_0(z_*^2)$, where $R_* \equiv (R-R_0)/\delta_R$, $z_* \equiv z/\delta_z$,

$N(R_*)$ is an odd periodic function R_* and $F_0(z_*^2) \to 0$ for $z_*^2 \to \infty$. When a seed magnetic field B_{z0} is present the Eq. (2) leads to find $\delta_R \sim R_0^{1/3}/k_0^{2/3}$ where $k_0^{-2} \equiv B_{z0}^2/(12\pi\rho_0\Omega_k^2)$ and ρ_0 is the peak density. Moreover, if we choose $\delta_z^2 = \delta_R^2/\varepsilon_*^2 < H_0^2 \equiv 2p_0/(\rho_0\Omega_k^2)$ we have $\delta_z \sim R_0^{1/2}/k_0^{1/2}$ and arrive at the following set of equations

$$\frac{1}{2}\frac{\partial}{\partial z_*}\left[P_* + \varepsilon_*^2\left(\frac{\partial y_*}{\partial z_*}\right)^2\right] + \varepsilon_*^2 D_* z_* + \frac{\partial^2 y_*}{\partial R_*^2}\frac{\partial y_*}{\partial z_*} = 0 \tag{3}$$

$$D_* y_* + \frac{1}{2}\frac{\partial}{\partial R}\left[P_* + \left(\frac{\partial y_*}{\partial z_*}\right)^2\right] + \varepsilon_*^2\frac{\partial^2 y_*}{\partial z_*^2}\frac{\partial y_*}{\partial R} = 0 \tag{4}$$

where $y_* \equiv \hat{\psi}/\tilde{\psi}_0$ where $\tilde{\psi}_0 = Max(\hat{\psi})$. In particular, the field $B_z \sim \hat{\psi}/(R_0\delta_R)$ due to the internal current is taken to be considerably larger than B_{z0}. Moreover, $P_* = P/P_0$, $p_0 = (B_{zMax})^2/8\pi$ and $D_* \equiv \rho/\rho_0$. The lowest order solution of Eqs. (3) and (4) produces the sequence of double rings configuration described in Ref. [3]. However, this solution has the property that both P_* and D_* vanish at $R_* = 0$ while the temperature has its (finite) maximum on this surface. Therefore, it is necessary to proceed to analyze the solution to next order in ε_*^2 which produces a component of $\hat{\rho}$ that is not vanishing at $R_* = 0$ and gives a different profile for the temperature represented by p_{tot}/ρ_{tot}.

III. Transport Controlled Configurations

We note that the toroidal component of the Lorentz force is $F_{M\phi} = (J_z B_R - J_R B_z)/c$, where $J_z = c/(4\pi) R\partial(B_\phi R)/\partial R$ and $J_R = -c/(4\pi)\partial B_\phi/\partial z$. Clearly, $F_{M\phi} \neq 0$ involves $B_\phi \neq 0$. Then, if a process for angular momentum transport is present and the resulting toroidal force is represented by $\Im_\phi$, the toroidal equilibrium equation can be written as $\Im_\phi = F_{M\phi}$, that is $\Im_\phi = \left[(1/R^2)\partial/\partial R\left(R^2 B_R B_\phi\right) + \partial/\partial z\left(B_z B_\phi\right)\right]/4\pi$.

The relevant poloidal currents are considered as driven by the angular momentum transport. In the present case the equilibrium is described

$$2R\Omega_k\rho(\delta\Omega) \simeq \frac{\partial}{\partial R}P_t + \frac{1}{4\pi R_0}\left(\frac{\partial^2\Psi}{\partial R^2} + \frac{\partial^2\Psi}{\partial z^2}\right)\left(\frac{\partial\Psi}{\partial R}\right), \tag{5}$$

$$0 \simeq \frac{\partial}{\partial z} P_t + \Omega_k^2 z \rho + \frac{1}{4\pi R_0}\left(\frac{\partial^2 \Psi}{\partial R^2} + \frac{\partial^2 \Psi}{\partial z^2}\right)\frac{\partial \Psi}{\partial z}, \quad (6)$$

$$\Im_\phi = \frac{1}{4\pi R_0^2}\left[\frac{\partial I}{\partial z}\frac{\partial \Psi}{\partial R} - \frac{\partial I}{\partial R}\frac{\partial \Psi}{\partial z}\right], \quad (7)$$

where $P_t \equiv p + B_\phi^2 / 8\pi$ and $B_\phi = I(\Psi, z)/R_0$. In particular, we may take

$$\Im_\phi \simeq -\rho R_0 D_\mu \frac{\partial^2}{\partial R^2}(\delta\Omega). \quad (8)$$

where D_μ is an effective viscous diffusion coefficient, and we may argue that $\Im_\phi$ is an even function of z and an odd function of $R - R_0$. This means that I and B_ϕ are odd functions of z. Moreover, J_z is an odd function of z, consistent with the symmetries of the currents associated with a jet that may emerge from the innermost double-ring element. The simplest case illustrating this point is that corresponding to the linearized perturbed theory [4] which starts from a disk that is immersed in a vertical magnetic field B_z. Then we can evaluate $\hat{J}_R$ from the equation

$\rho_0 R_0 D_\mu k_R^2 \delta\hat{\Omega} \simeq -\hat{J}_R B_z / c$, where $\delta\hat{\Omega} = \delta\tilde{\Omega} \sin\left[k_R (R - R_0)\right] \exp\left(-\bar{z}^2/2\right)$, $\bar{z}^2 = z^2 k_0 / H_0$, $k_R \simeq k_0, k_0 = \sqrt{3}\Omega_k / \mathrm{v}_A, H_0^2 \equiv (p_0/\rho_0)/\Omega_k^2$, p_0 and ρ_0 are the central pressure and density respectively. Consequently, we find $\hat{B}_\phi \simeq \tilde{B}_\phi \sin\left[k_0 (R - R_0)\right] \int_0^{\bar{z}} d\bar{z} \exp\left(-\bar{z}^2/2\right)$ where $\tilde{B}_\phi B_z / (4\pi\rho_0) = \delta\tilde{\Omega} D_\mu k_0^{3/2} H_0^{1/2} R_0$.

IV. Poloidal Velocities

In addition to the factors discussed in the previous sections it is important to take into account that poloidal velocities can have an important effect. Considering the constraints represented by, $\mathbf{E} + \mathbf{V} \times \mathbf{B}/c = 0$, $\nabla \cdot (\rho \mathbf{V}) = 0$ and $\mathbf{E} = -\nabla\phi$ we have, in general, $\mathbf{V} = \chi(\psi)\mathbf{B}/\rho(R,z) + \Omega(\psi) R \mathbf{e}_\phi$. Thus, $\mathrm{V}_\phi - \mathrm{V}_{\phi k}^0 = (\chi/\rho) B_\phi + \left[\Omega(\psi) R - \Omega_{k0} R_0\right]$ and, for $\Omega(\psi) = \Omega_{ko} + \delta\Omega(\psi_t)$, $\mathrm{V}_\phi - \mathrm{V}_{\phi k} \simeq \chi(B_\phi/\rho) + \Omega_{k0}(R - R_0) + \delta\Omega(\psi_t)$. Here, $\psi_t = \psi_t(R - R_0, z) = \psi_i + \psi_{ex}$, ψ_i corresponds to the surfaces produced by the internal currents and ψ_{ex} to the external field around $R = R_0$. In particular, the background field $B_{z0} \simeq (1/R_0) d\psi_{ex}/dR$ is represented by $\psi_{ex} = R_0 (R - R_0) B_{z0}$.

When $\chi = 0$ the component $\rho\left[\mathrm{V}_\phi^2/R - R\Omega_k^2(R)\right]$ of the radial equilibrium equation reduces to $\rho\Omega_{k0}\left[2R_0\delta\Omega(\psi_t)+3\Omega_{k0}(R-R_0)\right]$ and we may write $\delta\Omega(\psi_t)\simeq(d\Omega/d\psi_0)(\psi_i+\psi_{ex})$ where $R_0(d\Omega/d\psi_0)\psi_{ex}+(3/2)\Omega_{k0}(R-R_0)=0$. Clearly, when $\chi \neq 0$ and $4\pi\chi^2/\rho \ll 1$ the relevant contribution to the radial equation reduces to $\Omega_{k0}\left\{\rho\left[3\Omega_{k0}(R-R_0)+2R_0\delta\Omega(\psi_t)\right]+2\chi B_\phi\right\}$ and it has to be dealt with differently from the case where $\chi = 0$. Moreover, the relevant analysis involves again all three components (z, R, ϕ) of the total momentum conservation equation with special attention to the symmetries in $(R - R_0)$ and z of the considered solutions.

V. Mean Radial Inflow Velocity

In this case we consider a an average radial inflow velocity $\bar{\mathrm{V}}_R = \chi_0\,\bar{B}_R/\rho < 0$ that is much smaller than the plasma thermal velocity. When $\bar{B}_R$ is an even function of z and the background $\bar{B}_z$ field is negligible, it is easy to verify that a narrow set of "open" magnetic surfaces can develop, within the crystal configuration, on which particles can spiral inward toward the central object. An illustration of this is given in Fig. 1 where the ratio of B_R associated with the "crystal" configuration and $\bar{B}_R$ is about 12.5. In this case the axisymmetry of the magnetic configuration can be maintained. Another possibility which can be envisioned, but remains to be explored, is that the considered configuration be converted into a quasi tri-dimensional one and that the plasma flow toward the central object along pairs of spiraling O-lines and X-lines.

It is a pleasure to thank C. Crabtree and K. Tummel for their contribution to understanding the effects of poloidal flows and G. Bertin for his active interest in the issues discussed here.
*Sponsored in part by the U.S. Department of Energy.

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Fig. 1