*32nd EPS Conference on Plasma Phys. Tarragona, 27 June - 1 July 2005 ECA Vol.**29C**, P-5.119 (2005)*

Wake potential of a test charge using the stationary phase method

Michael A. Raadu, Muhammad Shafiq

Royal Institute of Technology, Alfvén Laboratory,

Division of Plasma Physics, Stockholm, Sweden

Abstract

The linear response of a dusty (complex) plasma to a moving test charge can be determined using an appropriate plasma dielectric function and a three dimensional Fourier analysis. Many analytical results have been found for a slowly moving test charge. For intermediate and large velocities numerical methods of integration are commonly used. However general asymptotic results valid at large distances can be found, using a combination of the residue calculus applied to the zeroes of the dielectric function and the method of stationary phase for integration over a wave vector component. The method can be expressed in terms of conditions on the group and phase velocity of waves in the reference frame of the moving test charge. In particular for a given radial direction the asymptotic response is determined by wave vectors for which the group velocity is directed radially outwards. The analysis is close to that due to Kelvin for ship waves in deep water. An essential difference is that in the present case three spatial dimensions are involved instead of only two, and also that the dispersion relation for plasma waves is more complicated.

Introduction

The field of dusty or complex plasmas has been one of the fastest growing areas of plasma physics research. In the test charge approach the plasma response to a localised perturbation is studied by calculating the electrostatic potential of a test charge moving through the plasma. The asymptotic form of the wake potential is here found using the method of stationary phase. The results can be related to the Kelvin ship wave analysis [1, 2].

Response to a moving test charge

For a test charge q_t moving with velocity $\mathbf{V_t}$ through a plasma, the general expression for the electrostatic potential is given by [3],

$$\phi = \frac{q_t}{8\pi^3\varepsilon_0}\int \frac{\exp[i\mathbf{K}\cdot\mathbf{r}]}{K^2 D(K,\omega)}\,\delta(\omega-\mathbf{K}\cdot\mathbf{V_t})\,d\omega\,d\mathbf{K} \tag{1}$$

where $D(K,\ \omega)$ is the dielectric constant function. For the case of a plasma with a cold dust component and including charging dynamics this is given by the expression,

$$D(K,\ \omega) = 1+\frac{K_1^2}{K^2}-\frac{\omega_{pd}^2}{\omega^2}+i\Delta \tag{2}$$

where $K_1 = \sqrt{K_e^2 + K_i^2}$, $\Delta = (K_{dch}^2/K^2)(\nu_0/\omega)$ and $K_{dch} = \sqrt{4\pi a_d n_{d0}\Omega_{\nu0}/\Omega_{u0}}$. The frequencies $\nu_0 = \Omega_{u0}$ and $\Omega_{\nu0}$ are as defined by Melandsø [4] and ω_{pd} is the dust plasma frequency. Here it is assumed that $\nu_0 << \omega$. Now we introduce cylindrical coordinates (ρ, ψ, z) where the z-axis is parallel to the test charge velocity $\mathbf{V}_t$. We then have $\mathbf{K}\cdot\mathbf{r} = K_\perp \rho \cos\psi + K_\| z$, where $K_\|$ ($\equiv K_z$) and $K_\perp$ ($\equiv K_\rho$) are the parallel and perpendicular components of the propagation vector $\mathbf{K}$. Carrying out the integrations over ψ and $K_\|$, Eq. (1) now gives

$$\phi = \frac{q_t/\varepsilon_0}{(2\pi)^2}\frac{1}{V_t}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{J_0(K_\perp\rho)\exp\left(i\frac{\omega}{V_t}z\right)}{\left(K_\perp^2+(\omega/V_t)^2\right)D\left(\sqrt{K_\perp^2+(\omega/V_t)^2},\omega\right)}K_\perp dK_\perp d\omega \tag{3}$$

where $J_0(K_\perp\rho)$ is the zero order Bessel function. (The integration over the δ-function replaces $K_\|$ by ω/V_t.) In order to solve the above Eq. (3), we look for the poles, i. e. when $D(K,\omega) = D\left(\sqrt{K_\perp^2+(\omega/V_t)^2},\omega\right) = 0$. Assuming that poles lie at $\omega = \omega_n$ where $\omega_n \equiv \omega_{nR} + i\omega_{nI}$ and $\omega_{nI} < 0$ (since the plasma is stable) we may write

$$D(K,\omega) \approx D(K,\omega_n) + \frac{\partial D}{\partial \omega}(\omega-\omega_n) + O(\omega-\omega_n)^2$$

where the first term on the right hand side is zero. Using the above equation and by residue theory we may write Eq. (3) for $z < 0$ as a sum over the poles as follows,

$$\phi = \frac{q_t/\varepsilon_0}{(2\pi)^2}\frac{2\pi i}{V_t}\sum_n\int_0^{\infty}\frac{J_0(K_\perp\rho)\exp\left(i\frac{\omega_n}{V_t}z\right)}{\left(K_\perp^2+(\omega_n/V_t)^2\right)\frac{\partial D}{\partial\omega}|_{\omega=\omega_n}}K_\perp dK_\perp \tag{4}$$

For $z > 0$ the response $\phi = 0$. Now the Bessel function $J_0(K_\perp\rho)$ takes the form $J_0(K_\perp\rho) = \sqrt{2/\pi K_\perp\rho}\left\{\cos\left(K_\perp\rho - \frac{\pi}{4}\right)\right\}$ for large argument [5]. Using this expression for the Bessel function, Eq. (4) may be written in simplified form as

$$\phi = \frac{q_t/\varepsilon_0}{2\pi}\frac{i}{V_t}\sum_n\sum_\pm\int_0^{\infty}\frac{\exp\left\{\pm i\left(K_\perp\rho \pm \frac{\omega_{nR}}{V_t}z - \frac{\pi}{4}\right)\right\}\exp[-\beta_n|z|]}{\sqrt{2\pi K_\perp\rho}\left(K_\perp^2+(\omega_n/V_t)^2\right)\frac{\partial D}{\partial\omega}|_{\omega=\omega_n}}K_\perp dK_\perp \tag{5}$$

where $\beta_n = -\omega_{nI}/V_t$ The terms in ϕ are seen to be spatially damped over lengths $1/\beta_n = V_t/|\omega_{nI}|$. For the case of a dusty plasma with dynamical charging Eq. (2) may be approximately solved for weak damping to show that $1/\beta > 2(K_1/K_{dch})^2 V_t/\nu_0$ so that for small K_{dch} the damping length is large compared to V_t/ν_0. Asymptotic approximations for Eq. (5) may now be found using the stationary phase method for large ρ and z.

Asymptotic form of the wake-potential

The method of stationary phase [6] may be applied to an integral of the form,

$$f(x,t) = \int \exp(i\eta(k))F(k)dk \tag{6}$$

with $\eta(k) = kx - \omega(k)t$, where $F(k)$ and $\eta(k)$ are real functions and both x and t are large. The contributions to the integral where there are rapid oscillations due to the changing phase η will average to zero. The only contributions come near stationary points where $\eta'(k) = 0$ that is where $\omega' = x/t$. By expanding $\eta(k)$ in a Taylor series about these stationary points $k = k_s$, the asymptotic form of the integral (for large x and t) can be obtained by summing the contributions from the neighbourhoods of each of these points.

$$f(x,t) \sim \sum_s \sqrt{\frac{2\pi}{-\omega''(k_s)t}}\, e^{i\pi/4} \exp(i\eta(k_s))\, F(k_s) \tag{7}$$

Comparing Eq. (6) with Eq. (5) we make the following identifications,

$$x \equiv \pm\rho;\ t \equiv \frac{z}{V_t};\ k \equiv K_\perp;\ \omega(k) \equiv \omega_{nR}(K_\perp)$$

The remaining terms in the integrand of Eq. (5) grouped together are equivalent to $F(k)$. The stationary phase condition is now,

$$\pm\rho + \frac{1}{V_t}\frac{d\,\omega_{nR}}{d\,K_\perp} z \equiv \eta'(k) = 0 \tag{8}$$

Using the condition for a pole ω_n that $D(|K|, \omega_n) = 0$, where $|K| = \sqrt{K_\perp^2 + (\omega_n/V_t)^2}$, the derivative $d\,\omega_{nR}/d\,K_\perp$ in the stationary phase condition Eq. (8) is found to be given by,

$$\frac{1}{V_t}\frac{d\,\omega_{nR}}{d\,K_\perp} = -\frac{V_g[K_\perp/\,|K|]}{V_g[\omega_{nR}/(V_t\,|K|)] - V_t} = -\frac{V'_{g\rho}}{V'_{gz}} \tag{9}$$

Here $\mathbf{V}_g$ is the group velocity in the plasma rest frame and $\mathbf{V}'_g \equiv \mathbf{V}_g - \mathbf{V}_t$ is the group velocity in a frame moving with the test charge velocity. From Eqs. (8) and (9) it follows that $V'_{g\rho}/V'_{gz} = |\rho|/z$ i.e. the group velocity in the moving frame must be in the direction of the radial vector from the test charge. This is the basic condition needed to find the wake pattern in the Kelvin analysis of ship waves [2].

To find the asymptotic expression for the response potential using Eq. (7) we use,

$$\omega''(k_s) \equiv \left[\frac{d^2\,\omega_{nR}}{d\,K_\perp^2}\right]_{K_\perp = K_{\perp s}} = -\frac{d}{d\,K_\perp}\left[\frac{V_t V_g[K_\perp/\,|K|]}{V_g[\omega_{nR}/(V_t\,|K|)] - V_t}\right]_{K_\perp = K_{\perp s}} \equiv \Lambda(K_{\perp s}(\theta)) \tag{10}$$

where $K_{\perp s}(\theta)$ are those values of $K_\perp$ that satisfy the stationary phase condition Eq. (8) for a given $\rho/z = \tan\theta$ where θ is the angle between the radial vector from the test charge and its velocity V_t. Finally the asymptotic form of Eq. (7) may now be written as,

$$\phi \sim \frac{i\,q_t}{2\pi\varepsilon_0}\sum_{n,\pm,s}\frac{\sqrt{K_{\perp s}}\,\exp\left\{\pm i\left(K_{\perp s}\rho \pm \frac{\omega_{nR}}{V_t}z - \frac{\pi}{4}\right)\right\}\exp[-\beta_n|z|]}{\sqrt{i\Lambda(K_{\perp s})\,V_t\,\rho\,z}\left(K_{\perp s}^2 + (\omega_n/V_t)^2\right)\left[\frac{\partial D}{\partial\omega}\right]_{\omega=\omega_n}} \tag{11}$$

Here it should be noted that several terms ($\omega_n, \omega_{nR}, \beta_n, \ldots$) depend implicitly on on $K_{\perp s}(\theta)$ and the angle θ ($\tan\theta = \rho/z$).

Discussion

In this paper, we have found analytical results for the wake potential of a moving test charge in a multi-component dusty plasma using the method of stationary phase. The derivation leads to the two conditions used in Kelvin ship wave analysis: (i) The group velocity $\mathbf{V}'_g$ in the moving frame is in the radial direction from the test charge. (ii) The component of the test charge velocity $\mathbf{V}_t$ in the direction of the wave vector is equal to the phase velocity V_p of the waves ($\mathbf{K}.\mathbf{V}_t = KV_p = \omega$). These two conditions together with a relation between the group and phase velocities are sufficient to determine the small scale structure of the wake [2]. In the case of dust acoustic waves $V_g \propto V_p^3$ (for ship waves $V_g = V_p/2$) and numerical results [7] confirm that the small scale wake structure is close to that given by such a modified Kelvin wave analysis. The stationary phase analysis performed here also in priciple gives the asymptotic slowly varying amplitude of the spatial oscillations, Eq. (11).

Acknowledgement

The authors would like to thank their colleagues at the Alfvén laboratory for useful discussions and suggestions. This work was partially supported by the Swedish Research Council.

References

[1] H. Lamb, *Hydrodynamics, Sixth Edition* (Cambridge University Press, Cambridge, 1932) pp 433-437

[2] G. B. Whitham *Linear and Nonlinear Waves* (Wiley, New York, 1974) pp 409-414

[3] L. Stenflo and M. Y. Yu, Phys. Scr. **8**, 301 (1973)

[4] F. Melandsø, T. Aslaksen and O. Havnes, Planetary Space Sci. **41**, 321 (1993)

[5] M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standard, 1964)

[6] I. N. Sneddon,*Encyclopaedic Dictionary of Mathematics for Engineers and Applied Scientists* (Pergamon Press, Oxford, 1976) p 630

[7] P.-O. Svensson, *Response from a dusty plasma to a moving test charge*, Royal Institute of Technology, Division of Plasma Physics Internal Report, ALP-2004-102, 28 pp. (2004).