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Neoclassical Radial Electric Field and Flows in a Collisional Tokamak

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Abstract

The Pfirsch-Schlüter short mean free path description of magnetized plasma in the drift ordering is revisited to determine the radial electric field in a collisional tokamak of arbitrary cross section. A new contribution from an explicitly collisional portion of the ion distribution function is evaluated.

1. Introduction

The neoclassical electric field in a tokamak is determined by the conservation of toroidal angular momentum. In the steady state in the absence of momentum sources and sinks, it is explicitly evaluated by the condition that the radial flux of toroidal angular momentum vanish. Of course, for an isothermal plasma there is no momentum transport since the plasma rotates rigidly. When the ion temperature varies, obtaining the electric field from the full viscosity in terms of density and temperature gradients is far more complicated. Standard drift kinetics cannot be used to obtain this full ion viscosity [1, 2]. Recently [2] we evaluated the radial electric field for arbitrary tokamak cross sections including spherical tori by employing this viscosity. In the appropriate limit the result obtained reduces to the large aspect ratio, concentric, circular flux surface expression found by Claassen and Gerhauser [3] for equal ion and electron temperatures. Just recently we have discovered that there is an additional contribution to the radial angular momentum flux that arises from an explicitly collisional piece of the gyrophase dependent ion distribution function. The new terms, which we evaluate here, tend to be small, but are formally the same order as other temperature gradient terms in the perpendicular collisional viscosity when parallel and perpendicular scale lengths are the same order and the poloidal and toroidal magnetic fields are comparable.

2. Collisional Portion of the Gyrophase Dependent Ion Distribution Function

In [1, 2], a ν/Ω correction to the gyroviscosity that is formally the same order as the perpendicular viscosity, was not retained, where ν and Ω are the ion-ion collision and gyro frequencies. This collisional term was neglected because it was only expected to contribute to the collisional ion heat flux that is most easily evaluated by a moment approach. This portion of f corresponds to the solution of

$$\Omega\partial\tilde{f}_\nu/\partial\varphi = -C(\tilde{f}_1), \qquad (1)$$

with C the linearized ion-ion collision operator and φ the gyrophase. The only non-vanishing contribution to the right side of Eq. (1) is from the following piece of $\tilde{f}_1$ [1]:

$$\tilde{f}_1 \rightarrow \Omega^{-1} f_0 x^2 \vec{v} \cdot \vec{b} \times \nabla \ell n T , \qquad (2)$$

where f_0 is a stationary Maxwellian, $x^2 = Mv^2/2T$, and $\vec{b} = \vec{B}/B$ is the unit vector along the magnetic field $\vec{B}$, with T the ion temperature and M the ion mass. Using the result from Appendix C of Ref. [4], the solution to the preceding equation is

$$\tilde{f}_\nu = \frac{\nu Q(x) f_0}{\Omega^2} \vec{v}_\perp \cdot \nabla \ell n T , \qquad (3)$$

where

$$Q = -\frac{3(2\pi)^{1/2}}{x}\left[\left(1-\frac{5}{2x^2}\right)E(x)+\frac{5}{2x^2}E'(x)\right], \qquad (4)$$

with $E(x) = 2\pi^{-1/2}\int_0^x dt \exp(-t^2)$ the error function and $E'(x)$ its derivative.

3. New Collisional Contribution to the Angular Momentum Flux

The collisional term $\tilde{f}_\nu$ contributes to the viscosity $\overleftrightarrow{\pi}$ through a term similar to the one that leads to the gyroviscous contribution $\overleftrightarrow{\pi}_g$ at lower order [5], namely,

$$\overleftrightarrow{K}_\nu \equiv \nabla \cdot [M\int d^3v \tilde{f}_\nu \vec{v}(\vec{v}\vec{v} - \overleftrightarrow{I} v^2/3)] + (\overleftrightarrow{I} - 3\vec{b}\vec{b})\vec{b} \cdot \{\nabla \cdot [(M/2)\int d^3v \tilde{f}_\nu \vec{v}(\vec{v}\vec{v} - \overleftrightarrow{I} v^2/3)]\} \cdot \vec{b}, \quad (5)$$

where as in [1] this new explicitly collisional viscosity $\overleftrightarrow{\pi}_\nu$ is determined from

$$\Omega(\overleftrightarrow{\pi}_\nu \times \vec{b} - \vec{b} \times \overleftrightarrow{\pi}_\nu) = \overleftrightarrow{K}_\nu . \qquad (6)$$

Notice that $\overleftrightarrow{K}_\nu$ is traceless with $\vec{b} \cdot \overleftrightarrow{K}_\nu \cdot \vec{b} = 0$ as required. The full viscosity then becomes

$$\overleftrightarrow{\pi} = M\int d^3v f(\vec{v}\vec{v} - \overleftrightarrow{I} v^2/3) = \overleftrightarrow{\pi}_{||} + \overleftrightarrow{\pi}_g + \overleftrightarrow{\pi}_\perp + \overleftrightarrow{\pi}_\nu , \qquad (7)$$

where $\overleftrightarrow{\pi}_{||}$ and $\overleftrightarrow{\pi}_\perp$ are the parallel and perpendicular collisional viscosities [1].

To determine $\overleftrightarrow{\pi}_\nu$ we dot Eq. (6) from both sides by $R^2\nabla\zeta$. Recalling that $\vec{B} = I\nabla\zeta + \nabla\zeta \times \nabla\psi$, we find upon flux surface averaging that

$$\langle R^2 \nabla\zeta \cdot \overleftrightarrow{\pi}_\nu \cdot \nabla\psi \rangle = (B/2\Omega)\langle R^4 \nabla\zeta \cdot \overleftrightarrow{K}_\nu \cdot \nabla\zeta \rangle , \qquad (8)$$

where $\langle ... \rangle$ denotes flux surface average and $\langle R^2 \nabla\zeta \cdot \overleftrightarrow{\pi} \cdot \nabla\psi \rangle$ the radial flux of toroidal angular momentum. Evaluating $\langle R^4 \nabla\zeta \cdot \overleftrightarrow{K}_\nu \cdot \nabla\zeta \rangle$ by the procedures used in [1] gives

$$\langle R^2 \nabla\zeta \cdot \overleftrightarrow{\pi}_\nu \cdot \nabla\psi \rangle = \qquad (9)$$

$$\left\langle \nabla \cdot \left[\frac{2cp\nu}{5e\Omega^2}\left(R^2 - \frac{I^2}{B^2}\right)\nabla T\right]\right\rangle + \frac{2cp\nu}{5e}\left\langle \left(R^2 - \frac{3I^2}{B^2}\right)\frac{\vec{\kappa}\cdot\nabla T}{\Omega^2} - \frac{1}{\Omega^2}\nabla_\perp T \cdot \nabla\left(2R^2 - \frac{I^2}{B^2}\right)\right\rangle ,$$

where $\vec{\kappa} = \vec{b} \cdot \nabla \vec{b}$, p = nT, and we used $\int_0^\infty dx x^6 Q(x) \exp(-x^2) = -3\pi^{1/2}/4$. Interestingly, in a collisional plasma, $\tilde{f}_\nu$ is the only portion of the second order in gyroradius correction to the Maxwellian, f_2, leading to non-vanishing terms in the moment giving $\overleftrightarrow{\pi}_\nu$.

4. Angular Momentum Transport in the Pfirsch-Schlüter Regime

If the new term is combined with the results given by Catto and Simakov [2] the complete expresion for the radial flux of toroidal angular momentum becomes

$$\langle R^2\nabla\zeta\cdot\vec{\pi}\cdot\nabla\psi\rangle = \langle R^2\nabla\zeta\cdot(\vec{\pi}_g+\vec{\pi}_\perp+\vec{\pi}_\nu)\cdot\nabla\psi\rangle, \tag{10}$$

where the radial electric field in the steady state in the absence of sources or sinks is found from $\langle R^2\nabla\zeta\cdot\vec{\pi}\cdot\nabla\psi\rangle = 0$. In effect the new collisional term leads to the replacement $\langle R^2\nabla\zeta\cdot\vec{\pi}_\perp\cdot\nabla\psi\rangle \Rightarrow \langle R^2\nabla\zeta\cdot(\vec{\pi}_\perp+\vec{\pi}_\nu)\cdot\nabla\psi\rangle$. The contribution from $\langle R^2\nabla\zeta\cdot\vec{\pi}_\nu\cdot\nabla\psi\rangle$ is the same order as other $dT/d\psi$ terms in $\langle R^2\nabla\zeta\cdot\vec{\pi}_\perp\cdot\nabla\psi\rangle$ for comparable perpendicular and parallel scale lengths, as might be the case in a collisional spherical tokamak. These $dT/d\psi$ terms are $1/q^2$ smaller than the neoclassical terms from $\langle R^2\nabla\zeta\cdot\vec{\pi}_g\cdot\nabla\psi\rangle$, and are negligible for a large aspect ratio tokamak or when perpendicular scale lengths are small compared to the major radius. Here q is the usual tokamak safety factor.

5. Discussion and Results

For a collisional or Pfirsch-Schlüter short mean free path ordering in which the plasma flow is sub-sonic [1], we find that there are two interesting limiting cases [2], since the terms in the gyroviscosity always turn out to be proportional to $dT/d\psi$. The first is the simpler case of an extremely up-down asymmetric tokamak (for example, just inside the separatrix of a strongly single null divertor configuration) for which the lowest order gyroviscous contribution does not vanish and must be balanced by the leading order collisional viscous effect to determine the radial electric field. The second case is the more complicated case of an up-down symmetric tokamak for which the lowest order gyroviscosity vanishes and so it must be evaluated to higher order and balanced by the lowest order collisional viscosity to determine the radial electric field. In practice both contributions must be retained, and standard drift kinetics cannot be used to obtain the results [2]. Neither result is altered by the new term evaluated here as long as $q^2 >> 1$ or if the perpendicular scale lengths are much smaller than the major radius.

The lowest order flow in a collisional tokamak is $\vec{V} = \omega(\psi)R^2\nabla\zeta + u(\psi)\vec{B}$ with $\omega = -c[\partial\Phi/\partial\psi + (en)^{-1}\partial p/\partial\psi]$ containing the electrostatic potential Φ and the Pfirsch-Schlüter result [6] given by $u \approx -(1.8cI/e\langle B^2\rangle)\partial T/\partial\psi$. If the temperature is a constant then we recover the radial ion Maxwell-Boltzmann response

$$\langle R^2\nabla\zeta\cdot\vec{\pi}\cdot\nabla\psi\rangle = -\frac{3c^2M^2\nu p}{10e^2}\left\langle \frac{R^2B_p^2}{B^2}\left(R^2+\frac{3I^2}{B^2}\right)\right\rangle\frac{\partial\omega}{\partial\psi} = 0, \tag{11}$$

with B_p the poloidal magnetic field. For a strongly up-down asymmetric tokamak with spatial ion temperature variation $\langle R^2\nabla\zeta\cdot\vec{\pi}\cdot\nabla\psi\rangle = 0$ yields [2]

$$\frac{d\omega}{d\psi} \approx -\frac{4IdT/d\psi}{3M\nu\langle B^2\rangle}\frac{\langle R^2\vec{B}\cdot\nabla\ell nB\rangle}{\left\langle R^2B_p^2B^{-4}\left(R^2B^2+3I^2\right)\right\rangle}. \tag{12}$$

For the up-down symmetric temperature varying case the general $q^2 >> 1$ result is

$$\frac{3\nu pB}{2\Omega}\left\langle \frac{R^2B_p^2}{B^2}\left(R^2+\frac{3I^2}{B^2}\right)\right\rangle \frac{d\omega}{d\psi} = F(\psi)\left\langle R^2\nabla_{\|}B\int\frac{d\theta}{\vec{B}\cdot\nabla\theta}\left(1-\frac{B^2}{\langle B^2\rangle}\right) - \left(R^2-\frac{3I^2}{B^2}\right)\left(1-\frac{B^2}{\langle B^2\rangle}\right)\right\rangle +$$

$$\frac{d}{d\psi}\left\langle\left(R^2+\frac{3I^2}{2B^2}\right)(\vec{q}\cdot\nabla\psi) - R^2\nabla_{\|}B\int\frac{d\theta}{\vec{B}\cdot\nabla\theta}[\vec{q}\cdot\nabla\psi - \langle\vec{q}\cdot\nabla\psi\rangle]\right\rangle - \left\langle\left(R^2+\frac{7I^2}{2B^2}\right)\frac{d(\vec{q}\cdot\nabla\psi)}{d\psi}\right\rangle, \quad (13)$$

where $F=0.7[T_e/(T_e+T)][\langle B^2\rangle\langle B^{-2}\rangle - 1]^{-1}\langle\vec{q}\cdot\nabla\psi\rangle d\ell nT/d\psi$ and

$$\vec{q}\cdot\nabla\psi = -(8I^2p\nu/5M\Omega^2)(1-B^2/\langle B^2\rangle)dT/d\psi. \quad (14)$$

The expressions retaining symmetric and asymmetric effects as given by (12) and (13) are not altered by the new collisional term evaluated here. In the large aspect ratio, concentric, circular flux surface limit we recover the result of Claassen and Gerhauser [3]:

$$(r/\Omega_0)d\omega/dr \approx -0.19[q^3\rho_0^2T_e/T^2(T_e+T)](dT/dr)^2, \quad (15)$$

where $B_t=B_0R_0/R$, $\Omega_0=eB_0/Mc$, $\rho_0 = v_i/\Omega_0$, and $v_i = (2T_i/M)^{1/2}$.

If we assume that the plasma current is in the direction of increasing toroidal angle (the $\nabla\zeta$ direction) to make ψ increase outward from the magnetic axis, then the sign of I depends on the toroidal magnetic field direction. Notice, the magnetic topology of a tokamak impacts the radial electric field and ion flow velocity through reversals in the direction of (i) the plasma current or B_p, (ii) the toroidal magnetic field or I, or (iii) both; or (iv) by changing from lower single null to upper single null operation ($\vec{B}\cdot\nabla \rightarrow -\vec{B}\cdot\nabla$).

Edge momentum relaxation is expected to be anomalous, however, the electric field in the relaxed state need not be anomalous and may be determined at least in part by neoclassical considerations. The preceding expressions make it clear that the shear in the electric field and thereby the ion flow is set by the ion temperature profile. This shear and its effect on flow is separate from the sheared zonal flow driven by turbulence that is also axisymmetric. The neoclassical contribution to the axisymmetric radial electric field is not normally retained in fluid or kinetic codes.

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