32nd EPS Conference on Plasma Phys. Tarragona, 27 June - 1 July 2005 ECA Vol.29C, P-5.039 (2005)

Heat and momentum transport in arbitrary mean-free path plasma with a Maxwellian lowest order distribution function

Andrei N. Simakov[1], Peter J. Catto[2]

[1] *Los Alamos National Laboratory, Los Alamos, NM 87544, USA*

[2] *MIT Plasma Science and Fusion Center, Cambridge, MA 02139, USA*

1. Introduction

We use the drift kinetic formalism of Hazeltine [1], as recently generalized by Simakov and Catto [2], to obtain expressions for the ion perpendicular viscosity as well as for the ion and electron parallel viscosities, gyroviscosities, and heat fluxes for arbitrary mean-free path plasmas. Electron perpendicular viscosity is small and usually of no interest. All the results are obtained in terms of a few velocity moments of the gyrophase independent leading order in gyroradius expansion correction to the lowest order distribution function, the latter being assumed a Maxwellian.

2. Orderings and assumptions

We consider a quasineutral magnetized electron-ion plasma and assume that

$$\delta \equiv (\rho/L_\perp) \sim k_\perp \rho \ll 1, \tag{1}$$

where ρ is the ion gyroradius, and $L_\perp$ and $k_\perp$ are the characteristic perpendicular equilibrium length scale and wave vector, respectively. We also assume that

$$\frac{\partial}{\partial t} \sim \delta^2 \Omega, \quad \boldsymbol{v} \cdot \boldsymbol{\nabla} \sim \frac{e}{M} \boldsymbol{E} \cdot \boldsymbol{\nabla}_v \sim C \sim \delta \Omega, \tag{2}$$

where Ω and M are the ion gyrofrequency and mass, respectively, e is the unit electric charge, $\boldsymbol{E}$ is the electric field, $\boldsymbol{v}$ is the velocity variable of the ion drift kinetic equation, and C is the ion collision operator. Assumptions similar to those given by Eq. (2) are used for the electron drift kinetic equation as well. Finally, we assume for simplicity that the leading order distribution functions of both electrons and ions are Maxwellians. As argued in Ref. [3], the last assumption usually holds for plasmas of interest to the magnetic fusion confined by magnetic fields with closed flux surfaces in the absence of strong external driving forces, such as neutral beams or radio-frequency waves. We remark that this assumption is not essential and can be relaxed.

3. Viscosities

We evaluate ion viscosity $\overleftrightarrow{\pi} \equiv M \int \mathrm{d}^3 v (\boldsymbol{vv} - v^2 \overleftrightarrow{I}/3) f$ by forming the $M\boldsymbol{vv}$ moment of the full ion kinetic equation:

$$\frac{\partial p}{\partial t} \overleftrightarrow{I} + \boldsymbol{\nabla} \cdot \left(M \int \mathrm{d}^3 v\, \boldsymbol{vvv} f \right) - en(\boldsymbol{EV} + \boldsymbol{VE}) \tag{3}$$
$$-\Omega(\overleftrightarrow{\pi} \times \hat{\boldsymbol{b}} - \hat{\boldsymbol{b}} \times \overleftrightarrow{\pi}) + \frac{2m(p - p_e)\nu_e}{M} \overleftrightarrow{I} \approx M \int \mathrm{d}^3 v\, \boldsymbol{vv} C_{ii}(f),$$

where p and p_e are the ion and electron pressures, respectively, n is the plasma density, $\boldsymbol{V}$ is the ion flow velocity, $\hat{\boldsymbol{b}}$ is the unit vector along the magnetic field $\boldsymbol{B}$, m is the electron mass, ν_e is a characteristic electron-ion collision frequency as defined by Braginskii, C_{ii} is the ion-ion collision operator, and we dropped several small terms that are of no importance for this calculation. Then, $\overleftrightarrow{\pi} = \overleftrightarrow{\pi}_{\parallel} + \overleftrightarrow{\pi}_g + \overleftrightarrow{\pi}_{\perp}$, where the parallel viscosity

$$\overleftrightarrow{\pi}_{\parallel} \approx (\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \overleftrightarrow{I}/3)(p_{\parallel} - p_{\perp}) = (\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \overleftrightarrow{I}/3)\, M \int \mathrm{d}^3 v\, (v_{\parallel}^2 - v_{\perp}^2/2) \bar{f}, \tag{4}$$

with $\bar{f}$ the gyrophase averaged ion distribution function. The gyroviscosity is found from

$$\overleftrightarrow{\pi}_g = \frac{1}{4\Omega} \left[\hat{\boldsymbol{b}} \times \overleftrightarrow{K}_g \cdot (\overleftrightarrow{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) - (\overleftrightarrow{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) \cdot \overleftrightarrow{K}_g \times \hat{\boldsymbol{b}} \right], \tag{5}$$
$$\overleftrightarrow{K}_g \equiv \boldsymbol{\nabla} \cdot \left(M \int \mathrm{d}^3 v\, \boldsymbol{vvv} f \right) - en(\boldsymbol{EV} + \boldsymbol{VE}),$$

and the perpendicular viscosity is obtained from

$$\overleftrightarrow{\pi}_{\perp} = \frac{1}{4\Omega} \left[\hat{\boldsymbol{b}} \times \overleftrightarrow{K}_{\perp} \cdot (\overleftrightarrow{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) - (\overleftrightarrow{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) \cdot \overleftrightarrow{K}_{\perp} \times \hat{\boldsymbol{b}} \right], \quad \overleftrightarrow{K}_{\perp} \equiv -M \int \mathrm{d}^3 v\, \boldsymbol{vv} C_{ii}(f). \tag{6}$$

Noticing that

$$\boldsymbol{\nabla} \cdot \left(M \int \mathrm{d}^3 v\, \boldsymbol{vvv} \bar{f} \right) = (2q_2 - 3q_1)(\hat{\boldsymbol{b}}\boldsymbol{\kappa} + \boldsymbol{\kappa}\hat{\boldsymbol{b}}) + \boldsymbol{\nabla} \cdot [(2q_2 - 3q_1)\hat{\boldsymbol{b}}]\, \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}$$
$$+ \boldsymbol{\nabla}(q_1 \hat{\boldsymbol{b}}) + [\boldsymbol{\nabla}(q_1 \hat{\boldsymbol{b}})]^{\mathrm{T}} + \boldsymbol{\nabla} \cdot (q_1 \hat{\boldsymbol{b}})\, \overleftrightarrow{I},$$

with $q_1 \equiv (M/2) \int \mathrm{d}^3 v\, v_{\parallel} v_{\perp}^2 \bar{f}_1$, $q_2 \equiv (M/2) \int \mathrm{d}^3 v\, v_{\parallel}^3 \bar{f}_1$, and $\boldsymbol{\kappa}$ the magnetic field curvature, and using the leading order gyrophase dependent ion distribution function $\tilde{f}_1$ [1] to obtain

$$\boldsymbol{\nabla} \cdot \left(M \int \mathrm{d}^3 v\, \boldsymbol{vvv} \tilde{f}_1 \right) \approx \boldsymbol{\nabla A} + (\boldsymbol{\nabla A})^{\mathrm{T}} + (\boldsymbol{\nabla} \cdot \boldsymbol{A})\, \overleftrightarrow{I}, \quad \boldsymbol{A} \equiv p\boldsymbol{V}_{\perp} + \frac{2}{5}\boldsymbol{q}_d,$$

with $\boldsymbol{V}_{\perp} \equiv \boldsymbol{v}_E + \boldsymbol{v}_d$, $\boldsymbol{v}_E \equiv c\boldsymbol{E} \times \hat{\boldsymbol{b}}/B$, $\boldsymbol{v}_d \equiv \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} p/(Mn\Omega)$, $\boldsymbol{q}_d \equiv (5p/2M\Omega)\hat{\boldsymbol{b}} \times \boldsymbol{\nabla} T$, and T the ion temperature, we eventually arrive at

$$\overleftrightarrow{\pi}_g = Mn \left[\boldsymbol{V}_{\parallel} \boldsymbol{v}_E + \frac{1}{4}(\boldsymbol{V}_{\perp}\boldsymbol{V}_{\perp} - \boldsymbol{V}_{\perp} \times \hat{\boldsymbol{b}}\, \boldsymbol{V}_{\perp} \times \hat{\boldsymbol{b}}) \right] + \frac{2q_2 - 3q_1}{\Omega} \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} \times \boldsymbol{\kappa} \tag{7}$$
$$+ \frac{1}{4\Omega} \hat{\boldsymbol{b}} \times (\overleftrightarrow{N} + \overleftrightarrow{N}^{\mathrm{T}}) \cdot (\overleftrightarrow{I} + 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) + \text{Transpose}, \quad \overleftrightarrow{N} \equiv p \boldsymbol{\nabla V}_{\perp} + \frac{2}{5} \boldsymbol{\nabla q}_d + \boldsymbol{\nabla}(q_1 \hat{\boldsymbol{b}}).$$

Expression (7) for gyroviscosity can be generalized for an arbitrary isotropic in velocity space [2] as well as for a simply arbitrary leading order distribution function [4].

Evaluation of perpendicular viscosity is more involved and reduces to evaluation of two integrals:

$$\overset{\leftrightarrow\ell}{K}_{\perp} \equiv -M\int \mathrm{d}^3v\, \boldsymbol{vv} C^{\ell}_{ii}(\tilde{f}_1+\tilde{f}_2) \quad \text{and} \quad \overset{\leftrightarrow n\ell}{K}_{\perp} \equiv -M\int \mathrm{d}^3v\, \boldsymbol{vv}[C^{n\ell}_{ii}(f_1,f_1)-C^{n\ell}_{ii}(\bar{f}_1,\bar{f}_1)],$$

where superscripts "ℓ" and "$n\ell$" stand for "linearized" and "non-linear" (bilinear), respectively, while subscripts 1 and 2 indicate the order in δ. Using the self-adjointness of $C^{\ell}_{ii}(f)$, noticing that [5] $C^{\ell}_{ii}(\boldsymbol{vv}f_M)=\nu_i F(x)\left(\boldsymbol{vv}-v^2/3\overset{\leftrightarrow}{I}\right)f_M$, with ν_i the characteristic ion collision frequency as defined by Braginskii, f_M a stationary Maxwellian, $F(x)$ a known scalar function, and $x\equiv\sqrt{Mv^2/2T}$, employing expressions for $\tilde{f}_1+\tilde{f}_2$ [1, 2], and performing the integration we can evaluate $\overset{\leftrightarrow\ell}{K}_{\perp}$ and ultimately obtain

$$\overset{\leftrightarrow\ell}{\pi}_{\perp} = -\frac{3\nu_i}{10\Omega^2}\left[\overset{\leftrightarrow}{W}+3\hat{\boldsymbol{b}}(\hat{\boldsymbol{b}}\cdot\overset{\leftrightarrow}{W})+3(\hat{\boldsymbol{b}}\cdot\overset{\leftrightarrow}{W})\hat{\boldsymbol{b}}+\frac{1}{2}(\overset{\leftrightarrow}{I}-15\hat{\boldsymbol{b}}\hat{\boldsymbol{b}})(\hat{\boldsymbol{b}}\cdot\overset{\leftrightarrow}{W}\cdot\hat{\boldsymbol{b}})-\frac{1}{2}(\overset{\leftrightarrow}{I}-\hat{\boldsymbol{b}}\hat{\boldsymbol{b}})\,\overset{\leftrightarrow}{W}:\overset{\leftrightarrow}{I}\right],$$

$$\overset{\leftrightarrow}{W} \equiv \boldsymbol{\nabla}\left(p\boldsymbol{V}_\perp+\frac{1}{10}\boldsymbol{q}_d-q_3\hat{\boldsymbol{b}}\right)-\boldsymbol{\nabla}\ln T\left(\frac{3}{4}p\boldsymbol{V}_\perp+\frac{9}{40}\boldsymbol{q}_d+q_4\hat{\boldsymbol{b}}\right)$$
$$-\frac{e\boldsymbol{E}}{T}\left(p\boldsymbol{V}_\perp-\frac{3}{10}\boldsymbol{q}_d-q_5\hat{\boldsymbol{b}}\right)-q_7\boldsymbol{\kappa}\hat{\boldsymbol{b}}+\text{Transpose}, \tag{8}$$

$$q_3\equiv\frac{5}{12}M\int\mathrm{d}^3v\,v_\parallel v_\perp^2F(x)\bar{f}_1,\quad q_4\equiv\frac{5}{24}M\int\mathrm{d}^3v\,v_\parallel v_\perp^2 xF'(x)\bar{f}_1,$$

$$q_5\equiv\frac{5}{6}M\int\mathrm{d}^3v\,v_\parallel\left[\frac{v_\perp^2}{4}F'(x)+\frac{T}{M}F(x)\right]\bar{f}_1,\quad q_6\equiv\frac{5}{6}M\int\mathrm{d}^3v\,v_\parallel\left(v_\parallel^2-\frac{3}{2}v_\perp^2\right)F(x)\bar{f}_1.$$

To evaluate the bilinear contribution $\overset{\leftrightarrow n\ell}{K}_{\perp}$ we use $\tilde{f}_1$ [1] to rewrite

$$\overset{\leftrightarrow n\ell}{K}_{\perp}=6\gamma M\int\mathrm{d}^3v(2\bar{f}_1+\tilde{f}_1)\left(\boldsymbol{V}_\perp\cdot\boldsymbol{\nabla}_v\boldsymbol{\nabla}_v\boldsymbol{\nabla}_vG_M+\frac{2T}{5pM}\boldsymbol{q}_d\cdot\boldsymbol{\nabla}_v\boldsymbol{\nabla}_v\boldsymbol{\nabla}_vH_M\right),$$

$$\gamma\equiv(3\sqrt{\pi}/2)(\nu_i/n)(T/M)^{3/2},\ G_M\equiv\int\mathrm{d}^3v'f_M(v')\,|\boldsymbol{v}-\boldsymbol{v}'|,\ H_M\equiv\int\mathrm{d}^3v'f_M(v')/|\boldsymbol{v}-\boldsymbol{v}'|.$$

Evaluating the integrals we finally arrive at

$$\overset{\leftrightarrow n\ell}{\pi}_{\perp}=-\frac{3Mn\nu_i}{10\Omega}\hat{\boldsymbol{b}}\times\boldsymbol{V}_\perp\boldsymbol{V}_\perp-\frac{9M\nu_i}{200pT\Omega}\hat{\boldsymbol{b}}\times\boldsymbol{q}_d\boldsymbol{q}_d+\frac{9M\nu_i}{100T\Omega}(\hat{\boldsymbol{b}}\times\boldsymbol{V}_\perp\boldsymbol{q}_d+\hat{\boldsymbol{b}}\times\boldsymbol{q}_d\boldsymbol{V}_\perp)$$
$$+\frac{q_7}{\Omega}\hat{\boldsymbol{b}}\times\boldsymbol{V}_\perp\hat{\boldsymbol{b}}+\frac{2q_8}{5p\Omega}\hat{\boldsymbol{b}}\times\boldsymbol{q}_d\hat{\boldsymbol{b}}+\text{Transpose}, \tag{9}$$

$$q_7\equiv12\gamma M\int\mathrm{d}^3v\,\frac{v_\parallel}{v}\left\{\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{1}{v}\frac{\mathrm{d}G_M}{\mathrm{d}v}\right)+\frac{v_\perp^2}{2}\frac{\mathrm{d}}{\mathrm{d}v}\left[\frac{1}{v}\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{1}{v}\frac{\mathrm{d}G_M}{\mathrm{d}v}\right)\right]\right\}\bar{f}_1,$$

$$q_8\equiv12\gamma M\int\mathrm{d}^3v\,\frac{v_\parallel}{v}\left\{\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{1}{v}\frac{\mathrm{d}H_M}{\mathrm{d}v}\right)+\frac{v_\perp^2}{2}\frac{\mathrm{d}}{\mathrm{d}v}\left[\frac{1}{v}\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{1}{v}\frac{\mathrm{d}H_M}{\mathrm{d}v}\right)\right]\right\}\bar{f}_1.$$

The ion perpendicular viscosity $\overset{\leftrightarrow}{\pi}_{\perp}=\overset{\leftrightarrow\ell}{\pi}_{\perp}+\overset{\leftrightarrow n\ell}{\pi}_{\perp}$.

It can be shown that Eqs. (4) and (7) also describe electron parallel viscosity and gyroviscosity when rewritten in terms of electron quantities. When $\bar{f}_1$ for short mean-free path plasma [6] is used to evaluate $q_1 - q_8$ the general expressions for gyro- and perpendicular viscosities recover the well-known short mean-free path limit [7].

4. Heat fluxes

The technique described in the previous section can be also used to evaluate the ion heat flux $\boldsymbol{q} \equiv \int \mathrm{d}^3v(Mv^2/2 - 5T/2)\boldsymbol{v}f$. Taking the parallel component we obtain

$$q_\| = q_1 + q_2 - 5pV_\|/2, \tag{10}$$

and similarly for the electrons. Evaluation of the diamagnetic and collisional perpendicular heat fluxes is easiest from the $(Mv^2/2 - 5T/2)\boldsymbol{v}$ moment of the ion kinetic equation, resulting in the standard Braginskii's expression

$$\boldsymbol{q}_\perp \approx \frac{5p}{2M\Omega}\hat{\boldsymbol{b}} \times \boldsymbol{\nabla}T - \frac{1}{\Omega}\hat{\boldsymbol{b}} \times \int \mathrm{d}^3v\,\frac{1}{2}Mv^2\boldsymbol{v}C_{ii}^\ell(\tilde{f}_1) = \frac{5p}{2M\Omega}\hat{\boldsymbol{b}} \times \boldsymbol{\nabla}T - \frac{2p\nu_i}{M\Omega^2}\boldsymbol{\nabla}_\perp T. \tag{11}$$

A similar procedure for electrons also gives the Braginskii's result for the electron diamagnetic and collisional perpendicular heat fluxes.

5. Conclusions

A drift formalism is used to describe heat and momentum transport in arbitrary mean-free path plasma with a Maxwellian lowest order distribution function. The results are obtained in terms of few velocity moments of $\bar{f}_1$. Ion viscosity is given by Eqs. (4), (7), (8), and (9), whereas the electron parallel viscosity and gyroviscosity are given by Eqs. (4) and (7) rewritten in terms of electron quantities. Electron and ion parallel heat fluxes are given by Eq. (10), whereas diamagnetic and collisional perpendicular heat fluxes are given by the standard Braginskii's expressions.

Acknowledgements

This research was supported by the USA Department of Energy.

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