*32nd EPS Conference on Plasma Phys. Tarragona, 27 June - 1 July 2005 ECA Vol.**29C**, P-4.121 (2005)*

Nonlinearly driven second harmonics of Alfvén Cascades

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Alfvén Cascades (ACs) have been observed in tokamaks in reversed shear operation, and have been theoretically explained as energetic particle or toroidicity induced shear Alfvén eigenmodes localised around the minimum q surface, [1][2]. In recent experiments in Alcator C-Mod, [3], measurements of density fluctuations with Phase Contrast Imaging through the plasma core show a second harmonic of the basic AC perturbation. The present work describes, assuming $\beta = 0$, poloidal mode number $m \gg 1$ and inverse aspect ratio $\varepsilon \ll 1$, the second harmonic density perturbation as driven by the first harmonic eigenmode through quadratic terms in the shear and compressional Alfvén wave equations. These quadratic terms vanish in a homogeneous straight magnetic field, but they can be important in the tokamak geometry, especially if there is also a density gradient.

Denote all quantities as $X_{\text{tot}} = X + \delta X$, where X is the equilibrium part, and the perturbed part δX can be written

$$\delta X = \sum_{l=1}^{\infty} \delta X_j \exp(-li\omega t) + \text{c.c.}, \tag{1}$$

and δX_1 dominates over all other harmonics. Write the plasma displacement, where the three degrees of freedom are represented by three scalar functions ξ, Φ and Ψ, as

$$\delta\mathbf{r} \equiv \xi\mathbf{b} + \frac{\mathbf{b}}{B} \times \nabla\Phi + \frac{1}{B}\nabla_\perp\Psi = \delta\mathbf{r}_\xi + \delta\mathbf{r}_\Phi + \delta\mathbf{r}_\Psi, \tag{2}$$

where B is the magnitude and $\mathbf{b}$ the direction of the equilibrium magnetic field. The continuity equation $\dot{\delta\rho} + \nabla\cdot((\rho+\delta\rho)(\dot{\delta\mathbf{r}} + (\dot{\delta\mathbf{r}}\cdot\nabla)\delta\mathbf{r})) = 0$, where dot denotes partial time derivative, gives the second harmonic density perturbation in terms of $\delta\mathbf{r}$ and thus in terms of ξ_1, Φ_1, Ψ_1, ξ_2, Φ_2 and Ψ_2 as

$$\begin{aligned} \delta\rho_2 &= \frac{1}{2}\nabla\cdot(\delta\mathbf{r}_1\nabla\cdot(\rho\delta\mathbf{r}_1) - \rho(\delta\mathbf{r}_1\cdot\nabla)\delta\mathbf{r}_1) - \nabla\cdot(\rho\delta\mathbf{r}_2) = \\ &= \delta\rho_{\xi_1^2} + \delta\rho_{\Phi_1^2} + \delta\rho_{\Psi_1^2} + \delta\rho_{\Phi_1\Psi_1} + \delta\rho_{\Phi_1\xi_1} + \delta\rho_{\Psi_1\xi_1} + \delta\rho_{\xi_2} + \delta\rho_{\Phi_2} + \delta\rho_{\Psi_2}. \end{aligned} \tag{3}$$

The first six of these contributions to the density perturbation are determined by quadratic terms of the first harmonic plasma displacement perturbations, and the last three contributions are determined by the second harmonic plasma displacement perturbations. The first harmonic coupling between the shear and the acoustic wave results in a slower frequency sweeping in the low frequency onset of the AC, but is unimportant for the main part of the cascade. For the main part one can then neglect the pressure terms in the momentum balance equation, which is used to determine the plasma displacement.

The first order (which is the first harmonic) momentum balance equation becomes

$$-4\pi\rho_0\omega^2\delta\mathbf{r}_1-(\nabla\times\mathbf{B})\times\delta\mathbf{B}_1-(\nabla\times\delta\mathbf{B}_1)\times\mathbf{B}=0. \tag{4}$$

and the second harmonic momentum balance equation becomes

$$\begin{aligned}&-4\pi\rho_0(2\omega)^2\delta\mathbf{r}_2-(\nabla\times\mathbf{B})\times\delta\mathbf{B}_{\mathrm{L}}-(\nabla\times\delta\mathbf{B}_{\mathrm{L}})\times\mathbf{B}=\\&=4\pi\omega^2\delta\mathbf{r}_1\delta\rho_1+4\pi3\omega^2\rho_0(\delta\mathbf{r}_1\cdot\nabla)\delta\mathbf{r}_1\\&+(\nabla\times\mathbf{B})\times\delta\mathbf{B}_{\mathrm{Q}}+(\nabla\times\delta\mathbf{B}_{\mathrm{Q}})\times\mathbf{B}+(\delta\mathbf{B}_1\cdot\nabla)\delta\mathbf{B}_1-\frac{1}{2}\nabla\delta\mathbf{B}_1^2,\end{aligned} \tag{5}$$

where we have put all the quadratic source terms on the right hand side and defined $\delta\mathbf{B}_{\mathrm{L}}=\nabla\times[\delta\mathbf{r}_2\times\mathbf{B}]$ and $\delta\mathbf{B}_{\mathrm{Q}}=1/2\ \nabla\times[((\delta\mathbf{r}_1\cdot\nabla)\delta\mathbf{r}_1)\times\mathbf{B}+\delta\mathbf{r}_1\times(\nabla\times[\delta\mathbf{r}_1\times\mathbf{B}])]$.

The vector component of the momentum balance equation along **b** gives the acoustic wave equation, the divergence of the perpendicular component gives the compressional wave equation, and the shear wave equation is obtained by taking the divergence of $\mathbf{b}/B\times$ the momentum balance equation. The first harmonic acoustic equation immediately yields $\xi_1=0$, because of the $\beta=0$ assumption, and this implies that $\delta\rho_{\xi_1^2}=\delta\rho_{\Psi_1\xi_1}=\delta\rho_{\Phi_1\xi_1}=0$. Using the fact that the AC frequency is below the toroidicity induced Alfvén eigenmode (TAE) frequency, so that $(\mathbf{B}\cdot\nabla)\sim k_\parallel\sim 1/(Rq)$, the first harmonic compressional wave equation gives the orderings

$$\Psi_1\sim\frac{\varepsilon^2}{m^2q^2}\Phi_1\ \Rightarrow\ \delta\rho_{\Phi_1\Psi_1}\sim\frac{m^2}{r^2R^2q^2}\frac{\rho}{B^2}\Phi_1^2\gg\delta\rho_{\Psi_1^2}, \tag{6}$$

Assume that the shear perturbations can be written $\Phi_j=\tilde{\Phi}_j(r)e^{i(jn\varphi-jm\theta)}$ where $d\tilde{\Phi}_j/dr\sim m/r$. In calculating the quadratic source terms, special care has to be taken to the fact that in a homogeneous straight magnetic field the shear contributions to the terms $4\pi\omega^2\rho_0(\delta\mathbf{r}_1\cdot\nabla)\delta\mathbf{r}_1$ and $(\delta\mathbf{B}_1\cdot\nabla)\delta\mathbf{B}_1$ in Eq. (5) cancel. The second harmonic acoustic equation gives the estimate $\xi_2\sim m^2\Phi_1^2/(r^2RqB^2)$, and the second harmonic compressional wave equation reduces to

$$\nabla\cdot\left(B^2\delta\mathbf{r}_{\Psi_2}\right)=-\frac{1}{2}\nabla\cdot\left(B^2[(\delta\mathbf{r}_{\Phi_1}\cdot\nabla)\delta\mathbf{r}_{\Phi_1}]_\perp\right)+\mathscr{O}\left(\frac{m^2\Phi_1^2}{r^2R^2q^2}\right). \tag{7}$$

From Eq. (3) one sees that $\delta\rho_{\Phi_1^2}+\delta\rho_{\Psi_2}=$

$$=\frac{1}{2}\nabla\cdot(\delta\mathbf{r}_{\Phi_1}\nabla\cdot(\rho\delta\mathbf{r}_{\Phi_1}))-\nabla\rho\cdot\left(\frac{1}{2}[(\delta\mathbf{r}_{\Phi_1}\cdot\nabla)\delta\mathbf{r}_{\Phi_1}]_\perp+\delta\mathbf{r}_{\Psi_2}\right)\sim\left(\frac{1}{R}+\frac{m\rho'}{\rho}\right)\frac{m^2\rho\Phi_1^2}{r^3B^2}, \tag{8}$$

which makes $\delta\rho_{\Phi_1\Psi_1}$ and $\delta\rho_{\xi_2}$ insignificant and $\delta\rho_{\Phi_1^2}+\delta\rho_{\Psi_2}$ thus only competes with $\delta\rho_{\Phi_2}$. To find Φ_2 one has to solve the shear wave equations, which can be simplified to

$$\frac{1}{r}\frac{d}{dr}\left(rD\frac{d\tilde{\Phi}_1}{dr}\right)-\frac{m^2}{r^2}\tilde{\Phi}_1 D+\frac{1}{r}\frac{d\overline{k_\parallel}^2}{dr}\tilde{\Phi}_1=0 \tag{9}$$

$$4\frac{1}{r}\frac{d}{dr}\left(rD\frac{d\tilde{\Phi}_2}{dr}\right)-16\frac{m^2}{r^2}\tilde{\Phi}_2 D+4\frac{1}{r}\frac{d\overline{k_\parallel}^2}{dr}\tilde{\Phi}_2=$$

$$=\frac{im}{rB}\left[\frac{dD}{dr}\left(\frac{1}{2}\frac{d^2(\Phi_1^2)}{dr^2}-2\frac{m^2}{r^2}\Phi_1^2\right)+3\frac{d}{dr}(AD)+\frac{d(\overline{k_\parallel^2})}{dr}A+\frac{1}{2}\frac{d\Phi_1^2}{dr}\frac{d^2D}{dr^2}\right], \tag{10}$$

where bar denotes flux surface average, $D=\omega^2/\bar{v}_A^2-\overline{k_\parallel}^2$, and $A=\tilde{\Phi}_1 d^2\tilde{\Phi}_1/dr^2-(d\tilde{\Phi}_1/dr)^2$. To derive Eq. (10), a flux surface average has been performed, since terms proportional to $\sin\theta$ or $\cos\theta$ only couples the first harmonic wave to the $2m\pm 1$ modes, which are not close to resonance. Taylor-expanding $\overline{k_\parallel}$ around the zero shear surface $r=r_0$ where the AC is situated, using the normalised coordinate $x\equiv(r-r_0)m/r_0$, enables us to rewrite Eqs. (9) and (10) as

$$\frac{d}{dx}(S+x^2)\frac{d\tilde{\Phi}_1}{dx}-(S+x^2)\tilde{\Phi}_1+Q\tilde{\Phi}_1=0 \tag{11}$$

$$4\frac{d}{dx}(S+x^2)\frac{d\tilde{\Phi}_2}{dx}-16(S+x^2)\tilde{\Phi}_2+4Q\tilde{\Phi}_2=$$

$$=\left(4x+3(S+x^2)\frac{d}{dx}\right)\left(T\frac{d^2T}{dx^2}-\left(\frac{dT}{dx}\right)^2\right)+x\left(\frac{d^2T^2}{dx^2}-4T^2\right)+\frac{dT^2}{dx}, \tag{12}$$

where $T^2(x)\equiv im^2\tilde{\Phi}_1^2/(r_0^2\bar{B})$ and

$$S\equiv\frac{2(\omega-\omega_0)\omega_0}{\bar{v}_A^2}\frac{mq_0}{r_0^2 q_0''}\frac{\overline{R}^2 q_0^2}{m-nq_0}. \tag{13}$$

The effects of hot ions and toroidicity has been added in the same way as in [2] through

$$Q\equiv\omega_0^2\frac{q_0^2\overline{R}^2}{\bar{v}_A^2(m-nq_0)}\frac{q_0}{r_0^2 q_0''}\left(\frac{\omega_{ch}}{\omega_0}\left(-\frac{r}{\rho}\frac{d\bar{\rho}_h}{dr}\right)_{r=r_0}+\frac{2m\varepsilon_0(\varepsilon_0+2\Delta_0')}{1-4(m-nq_0)^2}\right). \tag{14}$$

It was shown in [1] and [2] that solutions to Eq. (11) exist when $Q>1/4$. Through rewriting Eq. (11) for the function $G_1(x)=\tilde{\Phi}_1(x)\sqrt{S+x^2}$, and using a variational approach with the ansatz $G_1=A_1\exp(-x^2/(2a_1^2))$, one obtains for $Q=1$ that $a_1=1.247$ and $S=0.0983$, in agreement with the lowest order radial eigenmode solution in [1]. Solving Eq. (12) numerically (shown in Figure 1) then gives the estimate $\Phi_2\sim T^2\sim m^2\Phi_1^2/(r^2B)$ for $Q=1$, which implies that $\delta\rho_{\Phi_2}\gg\delta\rho_{\Phi_1\Psi_1}+\delta\rho_{\Phi_1^2}$ if $(\ln\rho)'\ll R^{-1}$, since

$$\delta\rho_{\Phi_2}=-\rho\nabla\Phi_2\cdot\left(\nabla\times\frac{\mathbf{b}}{B}\right)-\left(\frac{\mathbf{b}}{B}\times\nabla\Phi_2\right)\cdot\nabla\rho\sim\left(\frac{1}{R}+(\ln\rho)'\right)\frac{m^3}{r^3}\frac{\rho\Phi_1^2}{B^2}. \tag{15}$$

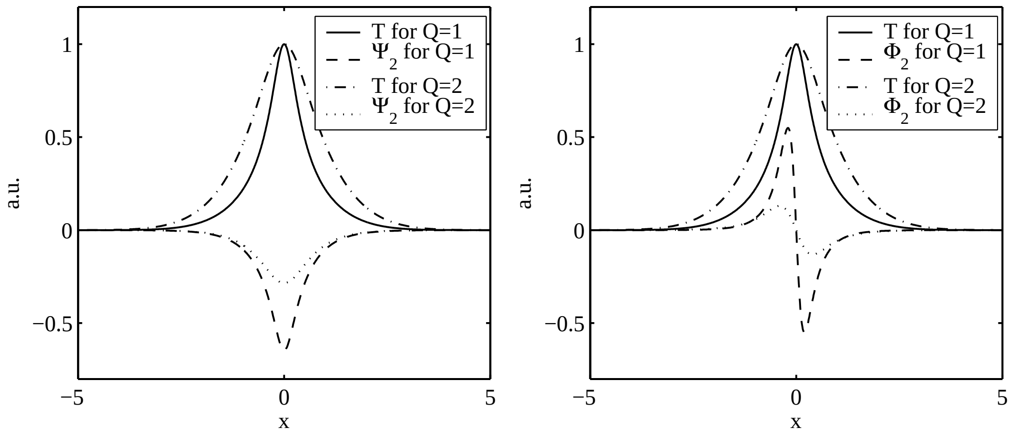

Figure 1: Solution to Eq. (7) (left) and Eq. (12) (right) for Q=1 and Q=2

Conclusions

The dominating part of the first harmonic density perturbation comes from the shear wave,

$$\delta\rho_1 = -\rho\nabla\Phi_1 \cdot \left(\nabla \times \frac{\mathbf{b}}{B}\right) - \left(\frac{\mathbf{b}}{B} \times \nabla\Phi_1\right) \cdot \nabla\rho \sim \left(\frac{1}{R} + (\ln\rho)'\right) \frac{m}{r}\frac{\rho\Phi_1}{B}, \tag{16}$$

so, the estimate for the ratio $\delta\rho_2/\delta\rho_1$, taking resonant enhancement into account, is for $Q = 1$

$$\frac{\delta\rho_2}{\delta\rho_1} \sim \frac{m^2}{r^2}\frac{\Phi_1}{B} \sim \frac{mq}{\varepsilon}\frac{|\delta\mathbf{B}_{\Phi_1}|}{B}. \tag{17}$$

The numerical solutions show that $\delta\rho_2$ decreases for increasing Q because of the widening of the first harmonic radial profile, and $\Phi_2 \sim 0.1T^2$ already for $Q = 2$. For high Q, the first term in Eq. (8) will then dominate $\delta\rho_2$, if not higher order radial eigenmodes of Φ_1 become excited.

In determining the accumulated phase shift for a laser beam passing the zero shear surface one has to note that the parts of $\delta\rho_2$ which are odd functions of x will give small contributions, and also that the terms containing $\nabla \times (\mathbf{b}/B)$ will have a dependence on the poloidal angle.

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