Effect of toroidal flow and flow shear on the quasi-interchange instability in tokamaks with weak magnetic shear

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Introduction Tokamak experiments in the “hybrid” scenario [1], as well as some experiments in spherical tokamaks [2], have q-values very close to unity in a wide area in the plasma core. Static, toroidal equilibria of this kind are susceptible to a pressure-driven, internal kink instability with an eigenfunction of “quasi-interchange” character [3]. In many experiments (especially in spherical tokamaks) the toroidal plasma rotation driven by the neutral beams is, however, quite substantial [1-2]. It is therefore of interest to examine the consequences of such plasma flows for the quasi-interchange (QI) instability. The present work investigates the effect of toroidal flow and, especially, toroidal flow shear on the QI mode using a compressible, ideal MHD description of the plasma. The work extends the analysis in Ref. [4], where weakly subsonic, rigid rotation was found to be strongly stabilizing.

Equilibrium We consider a toroidal plasma with circular cross section and large aspect ratio, \( \varepsilon = r/R_0 \ll 1 \), where \( r \) is the minor radius in the plasma and \( R_0 \) the major radius of the plasma center. Furthermore, we assume that the plasma rotates toroidally with a frequency of order \( \Omega \sim \omega_s \sim \epsilon \omega_A \), where \( \omega_s = (\Gamma p_0 / \rho_0)^{1/2} / R_0 \) is the sound frequency and \( \omega_A = B_0 / R_0 (\mu_0 \rho_0)^{1/2} \) the Alfvén frequency (\( \rho_0 \) and \( p_0 \) denote the plasma pressure and density, respectively, \( B_0 \) the toroidal magnetic field and \( \Gamma = 5/3 \) the adiabatic index). Assuming that the equilibrium magnetic surfaces are isothermal, the Shafranov shift \( \Delta \) is described by the equation [4] \( d\Delta/dr = -r(\beta_p + l/2)/R_0 \), with the plasma inductance \( l \) and the poloidal beta value \( \beta_p \) given by

\[
\begin{align*}
\beta_p (r) &= -\frac{2\mu_0 q^2 R_0}{r^4 B_0^2} \int_0^r \frac{d}{dr'} \left( p_0 + M_s^2 p_0 \right) dr', \\
\beta_p (r) &= -\frac{2\mu_0 q^2 R_0}{r^4 B_0^2} \int_0^r \frac{d}{dr'} \left( p_0 + M_s^2 p_0 \right) dr', \\
\end{align*}
\] (1a, b)

respectively. \( M_s = \left( \rho_0 \Omega^2 R_0^2 / 2 p_0 \right)^{1/2} \) in Eq. (1b) is the sonic Mach number. Thus, the rotation contributes to the Shafranov shift by increasing the “effective” poloidal beta value. When \( M_s \sim 1 \), the effect of the dynamical pressure \( M_s^2 p_0 = \rho_0 \Omega^2 R_0^2 / 2 \) is of the same order of magnitude as the effect of the static pressure. Here we specify the equilibrium profiles as follows: We let \( q = 1 \) in the region \( 0 \leq r \leq r_1 \) of weak magnetic shear, whereas in the edge region \( r_1 \leq r \leq a \) we assume that \( q(r) \) increases quadratically...
with $r$ from unity at $r = r_1$ up to $q_a = 4$ at the plasma edge $r = a$ [4]. Furthermore, $\rho_0(r)$ is assumed constant, and $p_0(r)$ and $\Omega(r)$ are chosen as

$$p_0(r) = p_0(0)(1 - r^2 / a^2), \quad \Omega(r) = \Omega_0 \left(1 - r^2 / r_\Omega^2\right)^{1/2}. \quad (2a, b)$$

This leads to a constant value of $\beta_p$ in the region $0 \leq r \leq r_1$. More specifically, we obtain the effective (total) beta value as

$$\beta_p = \beta_{p0} + \beta_{p\Omega},$$

where the static and dynamical parts of $\beta_p$ are given by $\beta_{p0} = \mu_0 p_0(0) R_0^2 / a^2 B_0^2$ and $\beta_{p\Omega} = \Omega_0^2 R_0^2 / 2\omega_0^2 r_\Omega^2$, respectively. Examples of the $q$- and $\Omega$-profiles used here are shown in Fig. 1. The rotation shear is described by the parameter $r_\Omega$, which is allowed to vary in the range $r_1 \leq r_\Omega \leq \infty$. We point out that the rotation profile is of importance only in the region of weak magnetic shear, so the dashed parts of the $\Omega$-curves in Fig. 1 are redundant for the QI stability problem looked at here. With other forms of the profiles of $\rho_0(r)$, $p_0(r)$ and $\Omega(r)$ we expect quantitative, but not qualitative, differences from the results of the stability analysis below.

**Stability analysis** Since the pressure- and rotation-profiles in Eqs. (2a, b) lead to a constant $E_p$ in the region of weak magnetic shear, the stability equation for the QI mode derived for a rigidly rotating plasma in Ref. [4] is valid also here. Thus, neglecting the second term in the inertia operator in Eq. (9b) in Ref. [4], the stability problem of the QI mode for the profiles in Eqs. (2a, b) can be formulated as

$$\int_0^1 \frac{\hat{r}^4 d\hat{r}}{Q(\hat{r}, \omega)} = \frac{\hat{r}_0^4}{\beta_p^2(3 + C)}, \quad \text{where } Q = -\hat{\omega}_D^2 + \hat{\Omega}^2 M_s^2 - \frac{4\hat{\omega}_D^2 + \hat{\Omega}^2 / 2}{\hat{\omega}_D^2 - \hat{\omega}_D^0} + 2\hat{\omega}_D^0 \hat{\omega}_D^2. \quad (3a, b)$$

Here, $C$ is a numerical constant related to the solution of the $m = 2$ side-band equation in the edge region [4], $\omega_D(r) \equiv \omega + (\Omega(r)$ is the Doppler-shifted mode frequency, and the normalized quantities in (3) are defined as $\hat{r} = r / a$, $\hat{\omega}_D = \omega_D / \epsilon_a \omega_A$, $\hat{\omega}_s = \omega_s / \epsilon_a \omega_A$, $\hat{\Omega} = \Omega / \epsilon_a \omega_A$, where $\epsilon_\alpha = a / R_0$. In addition to $\omega_D(r)$ and $\Omega(r)$, $r$-dependence of $Q(\hat{r}, \omega)$ comes from $\hat{\omega}_D^2 = \Gamma \beta_{p0} (1 - \hat{r}^2)$ and $M_s^2 = \hat{\Omega}^2 / 2\beta_{p0} (1 - \hat{r}^2) / (1 - \hat{r}^2) / 2\beta_{p0} (1 - \hat{r}^2)$.

In the present work, the eigenvalue equation (3) has been solved numerically, and the main features of the eigenfrequency $\omega$ are illustrated in Figs. 2-6. Two main regimes (and mechanisms) of rotational stabilization of the QI instability can be identified. Firstly, if the rotation is rigid or weakly sheared ($r_1 / r_\Omega \sim 2/3$ or smaller, see Fig. 1), the QI mode is stabilized by the Brunt-Väisälä (BV) mechanism explained in Ref. [4]. This is shown in Fig. 2 which illustrates the complex eigenvalue $\omega = \omega_r + i \omega_i$ (in normalized form) as a function of the (normalized) rotation frequency at the axis for the plasma parameters $r_1 = 0.5a$, $\beta_{p0} = 0.3$ and $r_\Omega = a$. Notice the splitting of $\omega_r$ (represented by the Doppler-shifted mode frequency at the magnetic axis) at the point of BV-stabilization. Secondly, if the rotation is strongly sheared ($r_1 / r_\Omega \sim 1$), a substantial reduction of the
growth rate takes place already for central rotation frequencies smaller than the rigid rotation frequency required for BV-stabilization. The dependence of $\omega_r$ and $\omega_\ell$ on the central rotation frequency in such a case is illustrated in Fig. 3 for a plasma with $r_1 = 0.2a$, $\beta_{p0} = 0.3$ and $r_\Omega = 0.225a$. Notice that there is no splitting of $\omega_\ell$ (dashed, blue curve) in this case (no BV effect) and that there is a limited range of $\Omega_0$ that gives a reduced growth rate $\omega_\ell$ (solid, red curve). Fig. 4 illustrates the growth rate and the loss of BV-stabilization (solid, red curves) in a plasma with the same parameters as in Fig. 2 as the shear increases from $r_\Omega/a = 1000$ ($\approx$ rigid rotation) up to $r_\Omega/a = 0.6$. As the flow shear becomes of order $r_1/r_\Omega \sim 1$, however, the growth rate is very strongly reduced, as shown by the dotted, green curve in the figure, calculated for $r_\Omega/a = 0.51$. This shear stabilization is, however, lost already for $r_\Omega/a > 0.52-0.53$. Fig. 5 illustrates the same effect in the case $r_1 = 0.2a$. Finally, the $\beta_{p0}$-dependence of the critical rotation frequency for BV-stabilization of the QI mode in the case of weak, or moderate, flow shear for both $r_1/a = 0.2$ and $r_1/a = 0.5$ is shown in Fig. 6. Notice that $\Omega_{0,\text{crit}}$ is normalized with the width $r_1$ of the region of low magnetic shear, leading to approximately similar stability boundaries for the two cases $r_1/a = 0.2$ (red) and $r_1/a = 0.5$ (blue). This is due to the scaling $\Omega_{0,\text{crit}} \sim r_1$ found in the case of rigid rotation in Ref. [4], which is seen to be (approximately) valid also if the rotation is weakly sheared. Furthermore, the $\beta_{p0}$-scaling $\Omega_{0,\text{crit}} \sim \beta_{p0}^{3/4}$ of the critical rotation frequency, strictly valid in the case of rigid rotation [4], is fairly accurate also for weakly sheared rotation.

**Conclusions** It has been shown that the Brunt-Väisälä mechanism, discussed in detail in Ref. [4], is capable of stabilizing the quasi-interchange instability in tokamaks with $q \approx 1$ in the plasma core also if the toroidal rotation is weakly sheared. Since the flow shear leads to an enhanced “effective” poloidal beta value $\beta_p$ (static plus dynamic), however, weakly sheared rotation requires somewhat higher central rotation frequencies for stabilization as compared with rigid rotation. At sufficiently strong flow shear, Brunt-Väisälä stabilization is in general not possible for reasonable central rotation frequencies. On the other hand, if the rotation is very strongly sheared ($r_1/r_\Omega \sim 1$), the shear itself is capable of reducing the growth rate of the instability to very small values.

Fig. 1 Examples of the rotation ($\Omega$) profile and the $q$-profile used in the numerical solution of the eigenvalue equation (3).

Fig. 2 The real (dashed) and imaginary(solid) parts of the eigenvalue $\omega$ as functions of $\Omega_0$. The figure illustrates the stabilization of the mode by the Brunt-Väisälä mechanism.

Fig. 3 The real (dashed) and imaginary(solid) parts of the eigenvalue $\omega$ as functions of $\Omega_0$. The figure illustrates the reduction of the growth rate by very strong flow shear.

Fig. 4 Growth rate as a function of $\Omega_0$ for various values of the flow-shear parameter $r_1/a$. The solid (red) curves with $r_1/a \geq 0.75$ correspond to Brunt-Väisälä stabilization and the dotted (green) curve to shear stabilization.

Fig. 5 Growth rate as a function of $\Omega_0$ for various values of the flow-shear parameter $r_1/a$. The solid (red) curves with $r_1/a \geq 0.4$ correspond to Brunt-Väisälä stabilization and the dotted (green) curve to shear stabilization.

Fig. 6 Critical rotation frequency as a function of $\beta_0$ for a few values of the flow-shear parameter $r_1/a$ and for $r_1/a = 0.2$ (red) and $r_1/a = 0.5$ (blue).