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Beam tracing solution of the weakly nonlinear Burgers' equation

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The beam tracing method [1-3] is an asymptotic technique for solving the *linear* wave equations relevant to the propagation of electrostatic [1] as well as electromagnetic waves in both isotropic and anisotropic (spatially non dispersive) plasmas [1, 2]. More recently, it has been successfully applied also to the description of short wavelength eigenmodes in a tokamak (microinstabilities) [3]. However, the beam tracing treatment of microinstabilities in the linear regime should be improved in order to account for nonlinear effects which are significant when the amplitude of an unstable eigenmode grows up.

The aim of this work is to study a particular *nonlinear* equation by means of the paraxial complex eikonal technique [4] which provides the proper generalization of the beam tracing method.

Specifically, let us consider a scalar velocity field $u(t, x, \kappa)$ in one spatial dimension depending on the large parameter $\kappa \gg 1$ and satisfying the equation

$$\partial u/\partial t + u\partial u/\partial x = \big(\mu\partial^2 u/\partial x^2 - \nu\partial^3 u/\partial x^3\big)/\kappa^2, \tag{1}$$

where μ and ν account for dissipation (viscosity) and dispersion, respectively. On rescaling such parameters according to $\mu/\kappa^2 \to \mu$ and $\nu/\kappa^2 \to \nu$, equation (1) amounts to the Korteweg-de Vries-Burgers equation which in the small dispersion limit, $\nu \to 0$, reduces to the Burgers' equation [5]. We are interested in solutions of the form

$$u(t, x, \kappa) = u_0(t, x) + \kappa^{-p} u_1(t, x, \kappa), \tag{2}$$

where u_1 is a perturbation of the background field u_0 with strength $1/\kappa^p$, $p \geq 1$ being an integer. The background field $u_0(t, x)$ is obtained as the limit $\kappa \to +\infty$ of a solution of the Korteweg-de Vries-Burgers equation (1) and, thus, it is expected to be a solution of the inviscid Burgers' equation [5] in an appropriate sense. We will consider two specific solutions:

$$\textit{(i)}\ u_0(t, x) = c_0 \quad \text{and} \quad \textit{(ii)}\ u_0(t, x) = c_0 x/(L + c_0 t).$$

with c_0 and L constant reference speed and length. In particular, the profile *(ii)* describes the central region of a shock-like solution [5].

Substituting (2) into (1) yields that the perturbation u_1 satisfies the equation

$$\frac{\partial u_1}{\partial t} + u_0\frac{\partial u_1}{\partial x} + (\partial u_0/\partial x)u_1 - \frac{\mu}{\kappa^2}\frac{\partial^2 u_1}{\partial x^2} + \frac{\nu}{\kappa^2}\frac{\partial^3 u_1}{\partial x^3} = \kappa^{-p} u_1 \frac{\partial u_1}{\partial x}, \tag{3}$$

for which we assume highly-oscillating initial conditions

$$u_1(t = 0, x, \kappa) = u_{\rm in}(x, \kappa) = \mathrm{Re}\big\{A_{\rm in}(x) e^{i\kappa S_{\rm in}(x)} e^{-\kappa\phi_{\rm in}(x)}\big\}. \tag{4}$$

In equation (4), $A_{\text{in}}(x)$ describes the amplitude modulation, whereas the short-scale variations ($\sim \kappa^{-1}$) of both the phase and the amplitude are accounted for by the complex eikonal function $\mathcal{S}_{\text{in}}(x) = S_{\text{in}}(x) + i\phi_{\text{in}}(x)$. Specifically, we will consider Gaussian wavepackets with curved phase-front so that, $A_{\text{in}}(x) = A_{\text{in}}$, $S_{\text{in}}(x) = \xi_{\text{in}}(x + x^2/2R_{\text{in}})$ and $\phi_{\text{in}}(x) = (x/w_{\text{in}})^2$, where A_{in} is the maximum value of the perturbation, ξ_{in} is the wavevector of the carrier wave (rescaled by κ), R_{in} the curvature radius of the phase front and w_{in} the e^{-1}-width of the Gaussian wavepacket.

Without the nonlinear term, i.e., with the right-hand-side equal to zero, equation (3) can be asymptotically solved by means of the complex eikonal ansatz [4], that is, $u_1(t,x,\kappa) \sim e^{i\kappa\mathcal{S}(t,x)} \sum_{j=0}^{+\infty} \kappa^{-j} A^{(j)}(t,x)$, with $\mathcal{S} = S + i\phi$, $\phi \geq 0$, the real-valued wavefield being obtained on taking the real part of the foregoing asymptotic solution. The effect of the nonlinear term on the right-hand-side of (3) is that of generating higher order harmonics ($\sim \exp(i\kappa n \mathcal{S})$ with n integer) at the order κ^{-p+1}. Being interested in the leading-order approximate solution, we will refer to the case $p \geq 2$ as the *linear case*, since the nonlinear term has no effect on the lowest order solution, in contrast to the *weakly nonlinear case* $p = 1$ for which the nonlinearity is significant even to the lowest order. In the latter case, the standard complex eikonal ansatz should be generalized in order to account for the nonlinear interaction of harmonics. The fully nonlinear case $p = 0$ is beyond the purpose of this work.

The linear case $p \geq 2$. For $p \geq 2$, one can still apply the complex eikonal ansatz provided that the asymptotic series is truncated, namely,

$$u_1(t,x,\kappa) \sim e^{i\kappa\mathcal{S}(t,x)} \sum_{j=0}^{p-2} \kappa^{-j} A^{(j)}(t,x) + \kappa^{-p+1}\, r_p(t,x,\kappa), \tag{5}$$

$r_p(t,x,\kappa)$ being the remainder of the expansion. It is worth noting that the asymptotic expansion (5) is useful as far as r_p is bounded in κ. On making use of (5) the leading order is determined by the complex eikonal equation

$$\frac{\partial \mathcal{S}}{\partial t} + u_0 \frac{\partial \mathcal{S}}{\partial x} - \nu \left(\frac{\partial \mathcal{S}}{\partial x}\right)^3 = 0, \tag{6}$$

together with the amplitude transport equation

$$\frac{\partial A}{\partial t} + \left[u_0 - 3\nu\Big(\frac{\partial \mathcal{S}}{\partial x}\Big)^2\right] \frac{\partial A}{\partial x} = -\left[\frac{\partial u_0}{\partial x} + \mu\Big(\frac{\partial \mathcal{S}}{\partial x}\Big)^2 - 3\nu \frac{\partial \mathcal{S}}{\partial x}\frac{\partial^2 \mathcal{S}}{\partial x^2}\right] A, \tag{7}$$

where the superscript (0) has been dropped for the amplitude. Equations (6) and (7) are readily solved on assuming that $\text{Im}\mathcal{S} = \phi = \big(x - x(t)\big)^2/w^2(t)$ where $x = x(t)$ is a reference curve in the (t,x)-space and $w(t)$ is the e^{-1}-width of the Gaussian envelope $e^{-\phi}$ centred at $x = x(t)$. As a consequence the wavefield is exponentially small for $\kappa \to +\infty$ far from the curve $x = x(t)$ and, thus, it is convenient to perform a Taylor expansion with respect to the spatial coordinate $x - x(t)$, referred to as the *paraxial expansion*. In particular, one gets $\mathcal{S}(t,x) = S_0(t) + \xi(t)(x - x(t)) + \frac{1}{2}\mathfrak{s}(t)\big(x - x(t)\big)^2 + \cdots$ and $A(t,x) = A_0(t) + \cdots$, where the dots stand for higher order terms which give a negligible contribution to the wavefield. Then equation (6) yields a set of ordinary differential equations for the curve $x(t)$, the wavevector $\xi(t)$ and for $\mathfrak{s}(t) = \xi(t)/R(t) + 2i/w^2(t)$, with $R(t)$ the curvature radius of the phase front; correspondingly, equation (7) yields an ordinary differential equation for the amplitude $A_0(t)$. Such equations can be solved analytically for the two considered profiles of the background velocity $u_0(t,x)$ and the solutions are shown in figure 1.

For the constant speed profile *(i)*, the reference curve amounts to the straight line $x(t) = (c_0 - 3\nu\xi_{\text{in}}^2)t$, with $\xi(t) = \xi_{\text{in}}$; therefore, the centre of the wavepacket travels with the speed $c_0 - 3\nu\xi_{\text{in}}^2$ which, in the small dispersion limit, reduces to the uniform background velocity c_0. Moreover, one has $S_0(t) = -2\nu\xi_{\text{in}}^3 t$, which vanishes in the small dispersion limit, and

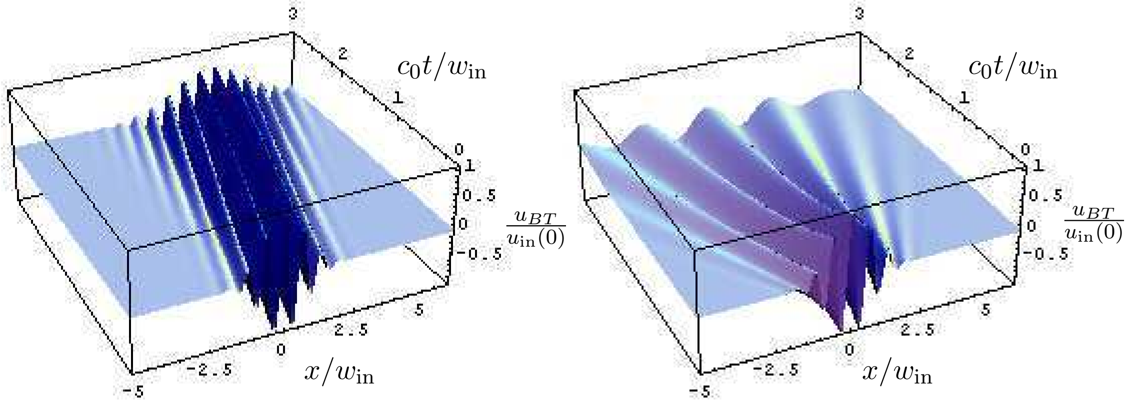
$$\mathfrak{s}(t) = \frac{\mathfrak{s}(0)}{1 - \mathfrak{s}(0)6\nu\xi_{\text{in}}t} = \frac{\xi_{\text{in}}}{R(t)} + i\frac{2}{w^2(t)}; \tag{8}$$

in particular, it is worth noting that both $R(t)$ and $w(t)$ are constant in the small dispersion limit. The corresponding solution for the amplitude is

$$A_0(t) = A_{\text{in}} e^{\frac{i}{2}\varphi(t)} \left(\frac{w_{\text{in}}}{w(t)}\right)^{\frac{1}{2}} e^{-\mu\xi_{\text{in}}^2 t}, \tag{9}$$

where the phase shift $\varphi(t) = \arg[1 - (6\nu\xi_{\text{in}}^2/R_{\text{in}})t + i(12\nu\xi_{\text{in}}/w_{\text{in}}^2)t]$ vanishes in the small dispersion limit. As a consequence of the scaling $A \sim w^{-1/2}$, the beam tracing solution $u_{BT} = Ae^{i\kappa\mathcal{S}}$ satisfies the conservation law $\int |u_{BT}|^2 dx \propto |A|^2 w$ =constant in the case of zero dissipation ($\mu = 0$); for $\mu \neq 0$ the exponential damping $e^{-\mu\xi_{\text{in}}^2 t}$ in (9) sets in. Let us note that, in the small dispersion and dissipation limit, $\mu, \nu \to 0$, the foregoing beam tracing solution reduces to $u_{BT}(t,x,\kappa) = u_{\text{in}}(x - c_0 t, \kappa)$ which is the exact solution of the right-propagating wave equation obtained from (3) in the limit $\mu, \nu \to 0$ and omitting the nonlinear term. Moreover, the beam tracing solution $u_{BT}\big(t, x(t)\big)$ evaluated on the reference curve $x = x(t)$ is exactly the same as the leading-order asymptotics, for $\kappa \to +\infty$, of the exact solution of (3) with the right-hand-side set to zero; the latter solution can be obtained by Fourier analysis and the limit $\kappa \to +\infty$ is performed by the stationary phase formula with complex phase function [6].

As for the linear time-dependent profile *(ii)* one finds that $\xi(t) = \xi_{\text{in}}/(1 + c_0 t/L)$, i.e., the spatial oscillations of the wavefield slow down following the slope of the background velocity profile; correspondingly, $x(t) = -\frac{3\nu\xi_{\text{in}}^2 L}{2c_0}[(1 + c_0 t/L) - (1 + c_0 t/L)^{-1}]$. In particular, the reference curve is tilted toward the negative x-axis as a consequence of dispersion and it reduces to the t-axis, i.e., $x(t) = 0$, for $\nu \to 0$. The quantities $S_0(t)$, $\mathfrak{s}(t)$ and $\varphi(t)$ can be obtained from those relevant to the case *(i)* by applying the



(i): $u_0(t,x) = c_0$. *(ii)*: $u_0(t,x) = c_0 x/(L + c_0 t)$.

Figure 1. The beam tracing solutions for the linear case ($p \geq 2$) corresponding to the profiles *(i)* and *(ii)*. The significant parameters are $\kappa = 10$, $\mu = 0$ (no dissipation), $\xi_{\text{in}} w_{\text{in}} = 1$ and $w_{\text{in}}/R_{\text{in}} = 0$ (initially flat phase front). The effects of dispersion are controlled by the parameter $3\nu\xi_{\text{in}}^2/c_0 = 0.15$ for the uniform profile *(i)* and by $3\nu\xi_{\text{in}} L/c_0 w_{\text{in}}^2 = 0.15$ for the varying profile *(ii)*. It is worth noting that for the case *(i)* the wavefield is localized around its reference curve even though the width of the wavepacket increases because of dispersion which plays the same role as diffraction for wavebeams. On the other hand, for the profile *(ii)* the oscillations slow down and the wavefield envelope spreads over the whole range of the plot so that the reference curve can no longer be noticed.

formal substitutions $t \to \frac{L}{2c_0}\big(1 - (1 + c_0 t/L)^{-2}\big)$ and $\mathfrak{s} \to \mathfrak{s}(1 + c_0 t/L)^2$. On the other hand, the amplitude takes the form

$$A_0(t) = A_{\text{in}} e^{\frac{i}{2}\varphi(t)} \left(\frac{w_{\text{in}}}{w(t)}\right)^{\frac{1}{2}} e^{-\mu \int_0^t \xi^2(t')dt'} (1 + c_0 t/L)^{-1/2}, \tag{10}$$

which exhibits the additional factor $(1+c_0 t/L)^{-1/2}$ accounting for the effects of $\partial u_0/\partial x$.

Discussion of the weakly nonlinear case $\boldsymbol{p = 1}$. The results for the case $p = 1$ are very preliminary. Difficulties are found in replacing (5) with an appropriate ansatz that accounts for the interaction of higher order harmonics. For instance, one can set $u_1(t,x,\kappa) = U\big(t, x, \kappa\mathcal{S}(t,x), \kappa\big)$, with $U(t,x,\theta,\kappa)$ a 2π-periodic function in θ, extended in the complex variable ζ with $\text{Re}\zeta = \theta$ and $\text{Im}\zeta \geq 0$. As a result one finds that multiple interacting solutions exist and they should be accounted for in the ansatz as, in view of the nonlinearity, the superposition principle is no longer valid. Up to now it appears that the proper ansatz for the considered equation including the dispersive term (which is crucial in the beam tracing method) is still lacking. This part of the work is in progress.

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