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Approximate relativistic dispersion relation for electron Bernstein waves in inhomogeneous plasma

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It is well known that the electrostatic approximation is quite accurate for the EBWs, so we start with the general electrostatic dispersion relation for plasma waves.

$$D = (\varepsilon_{11}N_\perp^2 + 2\varepsilon_{12}N_\perp N_\parallel + \varepsilon_{33}N_\parallel^2)/N^2 = 0 \quad (1)$$

where ε_{ik} are the dielectric tensor elements, $N_\perp \equiv k_\perp c/\omega$, $N_\parallel \equiv k_\parallel c/\omega$ are refractive indexes perpendicular and parallel to the local magnetic field direction, ω is the wave frequency, c is speed of light in vacuum and $N^2 = N_\perp^2 + N_\parallel^2$. Using known expressions for ε_{ik} in relativistic Maxwellian plasma, see for instance [1], and a Newberger sum rule for products of Bessel functions [2], the equation (1) can be written as

$$D = 1 + \frac{\upsilon\mu}{N^2}\{1 - W_1 - W_2\} = 0 \quad (2)$$

$$W_1 = -\frac{\pi q\mu}{2K_2(\mu)}\int_0^{+\infty} z_\perp dz_\perp \int_{-\infty}^{+\infty} \gamma\, e^{-\mu\gamma} J_p(x)\, N_p(x)\, dz_\parallel \quad (3)$$

$$W_2 = \frac{\pi q\mu}{2K_2(\mu)}\int_0^{+\infty} z_\perp dz_\perp \int_{-\infty}^{+\infty} \gamma\, e^{-\mu\gamma} \cot(\pi p)\, J_p^2(x)\, dz_\parallel \quad (4)$$

Here $z_\perp = P_\perp / m_e c$, $z_\parallel = P_\parallel / m_e c$ are normalised perpendicular and parallel impulses in phase space, $\gamma = \sqrt{1 + z_\perp^2 + z_\parallel^2}$ is the relativistic factor, $q = \omega/\omega_{ce}$, $\upsilon = \omega_{pe}^2/\omega^2$ where usual notations for the electron plasma and cyclotron frequencies are applied, $\mu = c^2/\mathrm{v}_{Te}^2$, $\mathrm{v}_{Te}^2 = T_e/m_e$, T_e is the electron temperature, $K_2(\mu)$ is the modified Hankel function of second order, $J_p(x)$ and $N_p(x)$ are the Bessel and Neumann functions of the order $p = q(\gamma - N_\parallel z_\parallel)$ with argument $x = qN_\perp z_\perp$. Principal contribution into the dispersion relation (2) makes the W_2 term (4), therefore we will concentrate below only on it because of lack of space in this four-page paper. For simplification of the equation (2) make use of the following inequalities typical for the EBW.

$$N_\perp \gg 1, \quad k_\perp \rho_{Te} = qN_\perp / \sqrt{\mu} \gg 1 \quad (5)$$

where ρ_{Te} is the Larmor radius of electrons at thermal velocity. Actually these inequalities mean that the main contribution to integral over $z_\perp$ in equation (2) comes from the region where the Bessel function argument in the integrands (3), (4) is large compared to unity. Using proper asymptotic expressions, like in the papers [3] and [4], for the Bessel functions at $x \gg 1,\ p \gg 1$ and regarding parameter $\mu \approx 500/T_e(keV) \gg 1$ as large, obtain

$$W_2 \approx \sqrt{\pi\mu/2N_\perp^2}\; e^{\mu-\tilde{\mu}}\, Q(\tilde{\mu},\tilde{q},N_\parallel) \qquad (6)$$

where new parameters $\tilde{\mu} = \mu\sqrt{1+1/N_\perp^2},\ \tilde{q} = q\sqrt{1+1/N_\perp^2}$ are introduced. Function Q can be presented in two alternative forms (7) or (8):

$$Q = \frac{\tilde{\mu}}{\pi\tilde{q}} \sum_{n=-\infty}^{+\infty} \Psi_n\left(\tilde{\mu},\tilde{q},N_\parallel\right) \qquad (7)$$

$$Q = \Lambda_0 + 2\sum_{m=1}^{m=+\infty} \Lambda_m \qquad (8)$$

Complex analytical functions Ψ_n of real arguments $\tilde{\mu},\ \tilde{q},\ N_\parallel$ are defined as follows:

$$\begin{aligned} \Psi_n &= \int_1^{+\infty} \frac{\exp[\tilde{\mu}\,(1-t)]\ t^2\ dt}{\sqrt{\left(1-N_\parallel^2\right)\left(t-t_1\right)\left(t-t_2\right)}} \quad \mathit{for} \quad N_\parallel^2 \neq 1 \\ \Psi_n &= \int_1^{+\infty} \frac{\exp[\tilde{\mu}\,(1-t)]\ t^2\ dt}{\sqrt{1+n^2/\tilde{q}^2 - 2\,t\,n/\tilde{q}}} \qquad \mathit{for} \quad N_\parallel^2 = 1 \end{aligned} \qquad (9)$$

Here integration path is along the real axis and square root under the integral is defined to be an unambiguous function on the complex plane cut along the negative real axis. The branch points $t_1,\ t_2$ of the square root are given by the following formula

$$t_{1,2} = \left[n/\tilde{q} \pm \sqrt{N_\parallel^2\left(N_\parallel^2 - 1 + n^2/\tilde{q}^2\right)}\right] \Big/ \left(1-N_\parallel^2\right) \qquad (10)$$

Complex summands of the representation (8) are defined by

$$\Lambda_m = -i\frac{e^{\tilde{\mu}\left(1-\sqrt{\chi_m}\right)}}{\chi_m^{5/2}}\left[\left(1-i\,m\,\lambda\right)^2\left(\chi_m + \frac{2\sqrt{\chi_m}}{\tilde{\mu}} + \frac{2}{\tilde{\mu}^2}\right) - m^2\lambda^2 N_\parallel^2\left(\frac{\sqrt{\chi_m}}{\tilde{\mu}} + \frac{1}{\tilde{\mu}^2}\right)\right] \qquad (11)$$

Here $\lambda \equiv 2\pi q/\mu$, $\chi_m \equiv \left(1-i\,m\,\lambda\right)^2 + m^2\lambda^2 N_\parallel^2$. Representation (7) is especially useful for analysis of analytical properties of the W_2, because it reveals contribution of each EC harmonic into the electron susceptibility. Alternative representation (8) is most suitable for numerical calculations when EC harmonics overlap. Imaginary parts of the Ψ_n can be found exactly using (9) and (10). There are three cases depending on $N_\parallel^2$ value. After long but simple calculations obtain for the case $N_\parallel^2 < 1$.

$$\mathrm{Im}\,Q = -\frac{\tilde{\mu}}{\tilde{q}\sqrt{1-N_{\|}^2}} \sum_{n>\tilde{q}\sqrt{1-N_{\|}^2}} e^{\tilde{\mu}(1-b)} \left\{ (a^2+b^2) I_0(\tilde{\mu}a) - a I_1(\tilde{\mu}a) \left[2b + \frac{1}{\tilde{\mu}} \right] \right\} \quad (12)$$

Here I_0 and I_1 are the modified Bessel functions and parameters a, b are defined as follows $b \equiv n / \tilde{q}\left(1-N_{\|}^2\right)$, $a \equiv |N_{\|}| \sqrt{N_{\|}^2 - 1 + n^2/\tilde{q}^2} \Big/ |1-N_{\|}^2|$. For $N_{\|}^2 > 1$ the imaginary part of the function Q becomes

$$\mathrm{Im}\,Q = -\frac{\tilde{\mu}}{\pi\tilde{q}\sqrt{N_{\|}^2 - 1}} \sum_{n=-\infty}^{+\infty} e^{\tilde{\mu}(1-b)} \left\{ (a^2+b^2) K_0(\tilde{\mu}a) + a K_1(\tilde{\mu}a) \left[2b + \frac{1}{\tilde{\mu}} \right] \right\} \quad (13)$$

where K_0 and K_1 are the modified Hankel functions. Parameters a, b entering formula (13) are the same as in the equation (12. The last possible case $N_{\|}^2 = 1$ is given by

$$\mathrm{Im}\,Q = -\sqrt{\frac{\tilde{\mu}}{2\pi\tilde{q}}} \sum_{n=1}^{+\infty} \frac{e^{\tilde{\mu}(1-d)}}{\sqrt{n}} \left\{ d^2 + \frac{3}{2\tilde{\mu}} \left[d + \frac{1}{\tilde{\mu}} \right] \right\}, \quad d \equiv \frac{1}{2}\left(\frac{\tilde{q}}{n} + \frac{n}{\tilde{q}} \right) \quad (14)$$

Approximate formulas (12)-(14) describe the imaginary part of the function Q for arbitrary $N_{\|}$ values. It should be noted that transition between regions $N_{\|}^2 < 1$ and $N_{\|}^2 > 1$ through the point $N_{\|}^2 = 1$ is smooth and continuous in these formulas. The anti-Hermitian (damping) part D^a of the dispersion relation (2) is proportional to the $\mathrm{Im}\,Q$. Corresponding exact formulas [5] contain infinite series of integrals that can be calculated only numerically. Figures 1 and 2 show comparisons between exact [5] and approximate (12) formulas for $N_{\|} = 0, N_{\|} = 0.2$ and parameters $N_{\perp} = 30,\ \mu = 100$ when $k_{\perp}\rho_{Te} = qN_{\perp} \big/ \sqrt{\mu} \approx 3$ and the initial suggestions (5) are satisfied, i.e., when our approximation is applicable. Approximate real part of the W_2 (6) with Q given by formulas (8), (11) and exact real part of the W_2 calculated directly from the double integral (4) for the same parameters as on Fig. 1 and Fig. 2 are shown on Fig. 3 and Fig. 4. Numerical modelling shows that the larger the $N_{\|}$ the better the agreement between exact and our approximate formulas for the W_2. When $N_{\|} \geq 0.3$ the approximate formulas in the region of their applicability (5) provide practically exact result (see example on Fig. 5, 6).

Acknowledgments

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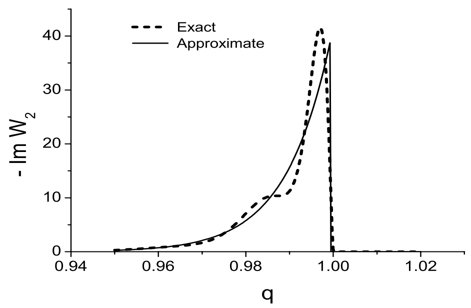

Figure 1 $N_{\|} = 0,\ \ N_{\perp} = 30,\ \ \mu = 100$

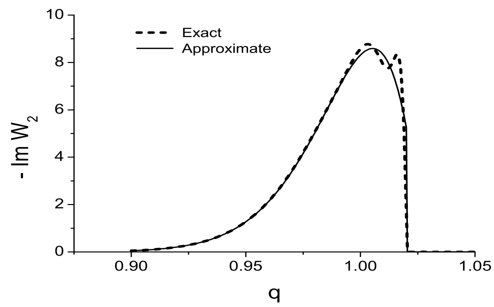

Figure 2 $N_{\|} = 0.2,\ \ N_{\perp} = 30,\ \ \mu = 100$

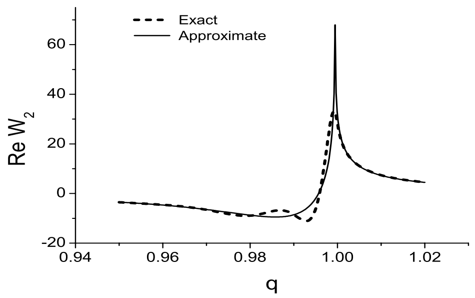

Figure 3 $N_{\|} = 0,\ \ N_{\perp} = 30,\ \ \mu = 100$

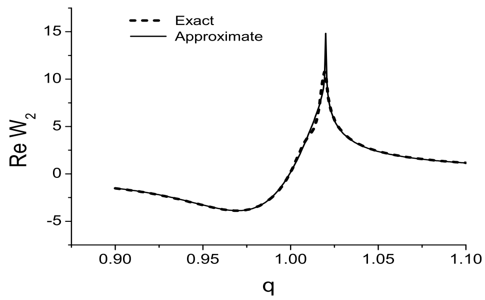

Figure 4 $N_{\|} = 0.2,\ \ N_{\perp} = 30,\ \ \mu = 100$

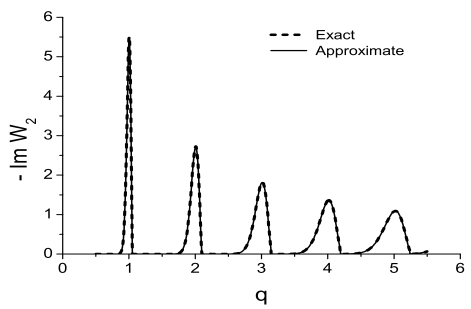

Figure 5 $N_{\|} = 0.3,\ \ N_{\perp} = 30,\ \ \mu = 100$

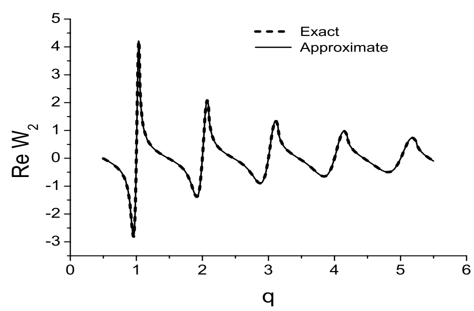

Figure 6 $N_{\|} = 0.3,\ \ N_{\perp} = 30,\ \ \mu = 100$