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Angular momentum coupling in tokamaks

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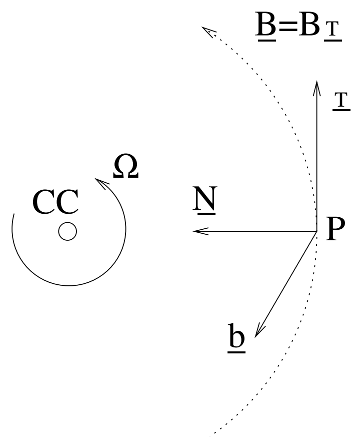
In the analysis of motion of a charged particle on a magnetic field line, Alfvén showed the existence of a force

$$\mathbf{f} = -\frac{mu_\perp^2 \kappa}{2} \mathbf{N}, \tag{1}$$

with $\mathbf{N}$ the first normal of the co-moving trihedral given by $(\boldsymbol{\tau}, \mathbf{N}, \mathbf{b})$ and κ the curvature. Here $\boldsymbol{\tau}$ is the unit vector along the magnetic field and $\mathbf{b}$ the binormal of the orthogonal system. Subscripts indicate direction with respect to the unit vector $\boldsymbol{\tau}$. In a reference frame located at the centre of curvature (CC) and rotating with an angular velocity Ω, a particle at point P and moving in a circular orbit, develops a centrifugal force $-m\Omega^2\rho\mathbf{N} = -(mu_\parallel^2/\rho)\mathbf{N}$ with ρ the distance from P to the origin of the reference frame. When combined with the force described by (1), this gives the total force



$$\mathbf{f}_t = -\frac{mu_\perp^2 \kappa}{2}\mathbf{N} - \frac{mu_\parallel^2}{\rho}\mathbf{N} = -m\kappa\left(\frac{u_\perp^2}{2} + u_\parallel^2\right)\mathbf{N}. \tag{2}$$

In a rotating reference system there exists also the Coriolis force, which will not be considered in this discussion. The force acting on a charge at point P produces the well known drift velocity

$$\mathbf{v}_d = \frac{\mathbf{f}_t \times \mathbf{B}}{qB^2} = \frac{1}{qB}\mathbf{f}_t \times \boldsymbol{\tau} = \frac{m\kappa}{qB}\left(\frac{u_\perp^2}{2} + u_\parallel^2\right)\mathbf{b}. \tag{3}$$

In a tokamak configuration the magnetic field lines lie on nested flux surfaces. With the toroidal component the dominant field, one can consider a reference system at the origin of the major radius rotating with an angular velocity $\Omega = \Omega(R)$.

In the derivation of expression (2) it was tacitly assumed that the angular velocity Ω is constant and therefore, the possible influence of the rotation of the reference system on the particle moving along a fiducial magnetic line is that of the centrifugal force only. Experimental evidence, however, especially during heating of plasmas in tokamaks with neutral beams and ion cyclotron resonance, shows that the plasma rotates in the toroidal direction with a varying angular rotation [1]. In this brief note we would like to investigate the effect of the relaxation of the ’rigid body’ assumption of the rotating reference frame on the motion of the particle.

Our point of departure is the Lagrangian

$$L = \frac{1}{2}mu^2 + m\mathbf{u}\cdot(\Omega\times\mathbf{r}) + \frac{1}{2}m(\Omega\times\mathbf{r})^2 \tag{4}$$

appropriate for such a system [2], where $\mathbf{u}$ is the velocity of the particle at a distance $\mathbf{r}$ in the rotating frame of reference. The gradients in terms of space and velocity are given by

$$\nabla L = m[(\nabla\Omega)\times\mathbf{r}]\cdot(\mathbf{u}+\Omega\times\mathbf{r}) + m(\mathbf{u}\times\Omega) + m\Omega\times(\mathbf{r}\times\Omega), \tag{5}$$

$$\frac{\partial L}{\partial \mathbf{u}} = m(\mathbf{u}+\Omega\times\mathbf{r}), \tag{6}$$

so that the Lagrangian equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{u}}\right) = \nabla L \tag{7}$$

becomes

$$m\frac{d\mathbf{u}}{dt} = \mathbf{f}_\Omega + \mathbf{f}_{\nabla\Omega}, \tag{8}$$

where we have assumed that the rotation is time independent, and with

$$\mathbf{f}_\Omega = m[\,2\mathbf{u}\times\Omega + \Omega\times(\mathbf{r}\times\Omega)\,], \tag{9}$$

$$\mathbf{f}_{\nabla\Omega} = m[(\nabla\Omega)\times\mathbf{r}]\cdot(\mathbf{v}+\Omega\times\mathbf{r}). \tag{10}$$

The two terms in $\mathbf{f}_\Omega$ are well known in the literature. We would like to remark that the second term $\Omega^2\mathbf{r} - (\Omega\cdot\mathbf{r})\Omega = \rho\Omega^2\hat{\mathbf{e}}_\rho$ is the centrifugal term $(u_\parallel^2/\rho)(-\mathbf{N})$ already included in (2). The first term is the Coriolis force which we neglect for the time being. The force $\mathbf{f}_{\nabla\Omega}$ owes its existence to the gradient in the angular velocity of the rotating system. Let $(\hat{\mathbf{i}},\hat{\mathbf{j}},\hat{\mathbf{k}})$ be a system of orthogonal unit vectors attached to the rotating reference frame, with $\Omega = \Omega\hat{\mathbf{k}}$. Then the force due to the gradient of the rotation of the medium can be written as

$$\mathbf{f}_{\nabla\Omega} = m\left[(xu_y - yu_x) + \Omega\left(x^2+y^2\right)\right]\nabla\Omega. \tag{11}$$

The angular momentum in the rotating reference frame is given by $\mathbf{M} = m\mathbf{r}\times(\mathbf{v}+\Omega\times\mathbf{r})$. Writing down its $\hat{\mathbf{k}}$ component, it is easy to show that (11) can be written as $\mathbf{f}_{\nabla\Omega} = M_z(\nabla\Omega)$. For a point with radius vector ρ in polar coordinates, with $\hat{\mathbf{e}}_\rho$ and $\hat{\mathbf{e}}_\phi$ the associated unit vectors, expression (11) becomes

$$\mathbf{f}_{\nabla\Omega} = m\rho^2\left(\dot{\phi}+\Omega\right)\nabla\Omega. \tag{12}$$

If the differential rotation has only a radial dependence, i.e. $\Omega = \Omega(\rho)$, this force becomes

$$\mathbf{f}_{\nabla\Omega} = m\rho^2\left(\dot{\phi}+\Omega\right)\frac{\partial\Omega}{\partial\rho}\hat{\mathbf{e}}_\rho = M_z\frac{\partial\Omega}{\partial\rho}\hat{\mathbf{e}}_\rho. \tag{13}$$

We observe that a radial force term appears due to the coupling of the angular momentum and the gradient of the differential rotation. In tokamak plasmas $\nabla\Omega < 0$ on the outboard side of the plasma column, so that the radial force is directed inward in these parts. We conclude that this force is a property of the Ω field – independent of the particle motion – as can be seen by eliminating the angular motion of some imaginary test particle ($\dot{\phi} = 0$).

For particles accelerated due to preferential heating, their average value over many rotations is expected to approach the value of the rotation of the background medium. In this case $\dot{\phi} = \Omega$ and

$$\mathbf{f}_{\nabla\Omega} = -2m\rho^2\Omega\,\frac{\partial\Omega}{\partial\rho}\,\mathbf{N}, \tag{14}$$

where $\mathbf{N}$ is directed in the opposite direction to that of the radial unit vector. Collecting all the forces considered so far, we find

$$\mathbf{f}_{tot} = -\frac{m}{\rho}\left(\frac{u_\perp^2}{2} + u_\parallel^2\right)\mathbf{N} - 2m\rho^2\Omega\,\frac{\partial\Omega}{\partial\rho}\,\mathbf{N}. \tag{15}$$

Finally, the drift velocity corresponding to the differential rotation is

$$\mathbf{v}_{\nabla\Omega} = -\frac{\rho^2}{\omega}\frac{\partial\Omega^2}{\partial\rho}\mathbf{b}, \tag{16}$$

with ω the cyclotron frequency of the particular species of particles under consideration, and $\mathbf{b}$ in the direction out of the plane of rotation. Figure 2(a) shows the radial profile of toroidal rotation of a plasma spun by neutral beam injection in the JET plasma [1], while Figure 2(b) shows the size of the drift velocity contribution due to the differential rotation.

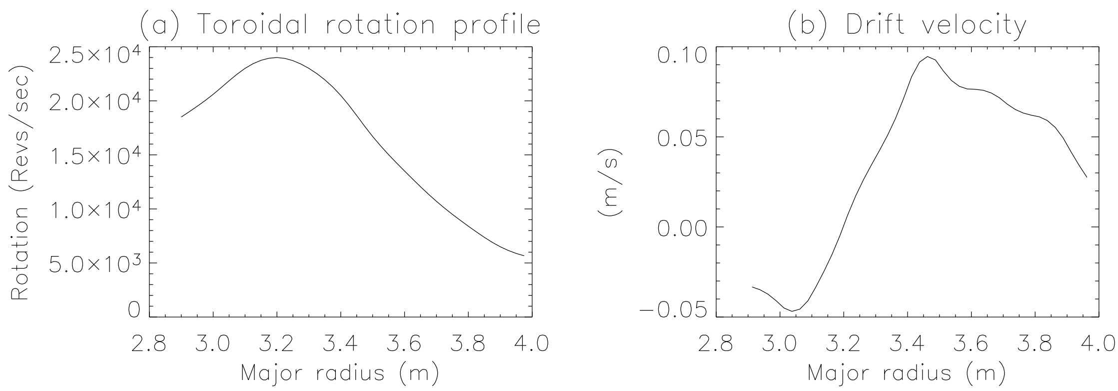


Figure 2: (a) Typical radial profile of toroidal rotation of a plasma spun by neutral beam injection in JET [1]; (b) Drift velocity (16) calculated for ions with $\omega = 10$MHz. The drift velocity of electrons has the opposite sign and is approximately 10^3 times smaller, since typical values for the electron cyclotron frequency in tokamaks are 60GHz $< \omega <$ 600GHz.

In the analysis the Coriolis force has been neglected. For rotation with a high frequency this is adequate, because the Coriolis contribution averages out to zero over many rotations. In the case of a low rotation frequency, it does make a contribution to the guiding centre drift. We would also like to emphasize the inability of the Coriolis force to do work, in contradistinction to the centrifugal force and $\mathbf{f}_{\nabla\Omega}$. When the Coriolis force is retained in (9), we obtain

$$\mathbf{f}_{\Omega} = -2m\Omega\dot{\rho}\hat{\mathbf{e}}_{\phi} + m\rho\Omega\left(2\dot{\phi}+\Omega\right)\hat{\mathbf{e}}_{\rho} \tag{17}$$

in plane polar coordinates, so that the force due to the Lagrangian (4) is

$$\mathbf{f}_{L} = -2m\Omega\dot{\rho}\hat{\mathbf{e}}_{\phi} + m\rho\Omega\left(2\dot{\phi}+\Omega\right)\hat{\mathbf{e}}_{\rho} + m\rho^2\left(\dot{\phi}+\Omega\right)\nabla\Omega. \tag{18}$$

This force has its origin in the dynamics of the particle. If the particle carries a charge, the Lorentz force comes into play and (1) should be added to the above. Under rotational symmetry (18) becomes

$$\mathbf{f}_{L} = -2m\Omega\dot{\rho}\hat{\mathbf{e}}_{\phi} + m\rho\left[\Omega\left(2\dot{\phi}+\Omega\right)+\rho\left(\dot{\phi}+\Omega\right)\frac{\partial\Omega}{\partial\rho}\right]\hat{\mathbf{e}}_{\rho}, \tag{19}$$

so that the guiding centre drift associated with it is

$$\mathbf{v}_d = \frac{\mathbf{f}_L\times\left(B\hat{\mathbf{e}}_{\phi}\right)}{qB^2} = \frac{\rho}{\omega}\left[\Omega\left(2\dot{\phi}+\Omega\right)+\rho\left(\dot{\phi}+\Omega\right)\frac{\partial\Omega}{\partial\rho}\right]\hat{\mathbf{e}}_{z}, \tag{20}$$

in polar coordinates. Here ω is the cyclotron frequency of the particles under consideration and we have used the fact that the toroidal field component is the dominant magnetic field in a tokamak. The Coriolis force, although it should be excluded in the evaluation of the work done by the acting forces, is capable of producing an excursion from an isoflux surface. If the plasma is frozen to the rotating reference frame, expressions (19) and (20) simplify to

$$\mathbf{f}_{L} = -2m\Omega\dot{\rho}\hat{\mathbf{e}}_{\phi} + m\rho\Omega\left(3\Omega+2\rho\frac{\partial\Omega}{\partial\rho}\right)\hat{\mathbf{e}}_{\rho}, \tag{21}$$

$$\mathbf{v}_d = \frac{\rho\Omega}{\omega}\left(3\Omega+2\rho\frac{\partial\Omega}{\partial\rho}\right)\hat{\mathbf{e}}_{z}. \tag{22}$$

The observed differential rotation in tokamaks has lead us to reconsider the dynamical problem of motion in a differentially rotating reference system. The inclusion of the radial variation of the angular rotation in the derivation of the equations of motion has produced a force term that couples the gradient of the angular velocity with the angular momentum. This force term is a property of the Ω field, so that the results should be valid not only for laboratory plasmas, but also for astrophysical plasmas where differential rotation is present.

References

[1] Wesson J., 2004, *Tokamaks*, (Oxford Science Publications, Oxford), par. 3.13.

[2] Landau L.D., Lifshitz E.M., 1975, *The Classical Theory of Fields*, (Butterworth-Heinemann Ltd., Oxford), par. 22.