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Statistical properties of a turbulent plasma of vortices interacting with random waves

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Statistical intermittency of a turbulent plasma appears in the regime where structures are generated at random, persist for a certain time and are destroyed by perturbations. This regime is supposed to be reached at stationarity in drift wave turbulence when there is a competition between different space scales. These scales range from the radially extended eddies of the ion temperature gradient (ITG) driven modes to the intermediate $\sqrt{L_n \rho_s}$ interval where the scalar nonlinearity is dominant over the vectorial one and with random fast decays to few ρ_s scales where robust vortex-like structures are generated. The analytical model we propose here is essentially *nonperturbative* in the sense that the structures are fully represented and we take into account their particular analytical expression (or a reasonable approximation). In usual perturbative approaches (including renormalization) the zero-perturbation state is either the equilibrium (zero fluctuations) or the linear waves and the perturbation develops beyond this state. In the present treatment the zero-perturbation state consists of an ensemble of vortices (with explicit expression) together with a Gaussian ensemble of waves and we extend the integrations to include configurations which account for their interactions. It is in the spirit of the semiclassical methods, largely used in field theory. The total fluctuating field is composed of vortices and random nonlinear waves $\varphi(x,y) = \sum_{a=1}^{N} \phi_s^{(a)}(x,y) + \phi(x,y)$ and it is the solution of the electrostatic $2D$ drift wave equations with dominant scalar nonlinearity. In the present treatment we take the static form of the equation $\nabla_\perp^2 \varphi - \alpha\varphi - \beta\varphi^2 = 0$ and the exact solution is $\varphi_s(y,t;y_0,u) = -3\left(\frac{u}{v_d} - 1\right) \sec\mathrm{h}^2 \left[\frac{1}{2\rho_s}\left(1 - \frac{v_d}{u}\right)^{1/2}(y - y_0 - ut)\right]$ where v_d and u are the diamagnetic and the plasma flow velocities, $\alpha = \frac{1}{\rho_s^2}\left(1 - \frac{v_d}{u}\right)$, $\beta = \frac{c_s^2}{2u^2}\frac{\partial}{\partial x}\left(\frac{1}{L_n}\right)$. We calculate the generating functional for the irreducible correlations of this field by integrating over the space of statistical ensemble of realizations. For this we first represent this space of functions as a direct product of two pieces: (1) one for the vortices which are perturbed by the presence of the random waves; and (2) one for the random waves scattered by randomly placed vortices and also interacting nonlinearly : $Z = Z_{V\phi} Z_{\phi V}$. Therefore in the first factor the main field is the vortices (V) and the turbulent waves (ϕ) is the perturbing factor, while in the second the nonlinear waves are the main field and the vortices are the perturbation. The two factors are connected by the presence of an external excitation, the same in both.

For the first part of the generating functional we have a dilute gas of weakly interacting vortices

$$\begin{aligned} Z_V &= \frac{1}{N!}\left(\prod_{j=1}^{N} Z_V^{(0)}\right) \times \sum_{\{\alpha\}} \int \left(\prod_{a=1}^{N} \frac{1}{A} d\mathbf{R}^{(a)}\right) \\ &\times \exp\left[-\pi \sum_{\substack{a=1 \\ a>b}}^{N} \sum_{b=1}^{N} \int dxdy \int dx'dy' \rho_\omega^{(a)}(x,y) K_0\left(\frac{\left|\mathbf{R}^{(a)}-\mathbf{R}^{(b)}\right|}{\rho_s}\right) \rho_\omega^{(b)}(x',y')\right] \end{aligned}$$

where $\rho_\omega^{(a)}(x,y)$ is the density of vorticity and the sum is over the configurations of random positions in plane of vortices. The turbulent field acts upon any individual vortex inducing a fluctuation of its shape which is expressed by the factors $Z_V^{(0)}$. We have the normalization factor

$$Z_V[J_\omega=0] = \exp\left[Z_V^{(0)}\right] p^{-1}\sqrt{\pi}\left[\frac{1}{8\pi^2}\rho_s^2\left(k^2+\frac{1}{\rho_s^2}-\frac{4\pi^2}{A}Z_V^{(0)}\right)\right]^{-1/2}$$

where A is the area of the region in plane. Since $Z_V^{(0)}$ is also subject to the external excitation, we will have

$$\begin{aligned} \langle \phi_V(x,y)\,\phi_V(x',y')\rangle &= \frac{1}{Z_V[j=0]}\left[\left(\frac{\delta^2 Z_V}{\delta J(x,y)\,\delta J(x',y')}\right)_{vort}\right. \\ &\left. +\frac{\delta^2 Z_V}{\delta\left(Z_V^{(0)}\right)^2}\frac{\delta Z_V^{(0)}}{\delta J(x,y)}\frac{\delta Z_V^{(0)}}{\delta J(x',y')} + \frac{\delta Z_V}{\delta Z_V^{(0)}}\frac{\delta^2 Z_V^{(0)}}{\delta J(x,y)\,\delta J(x',y')}\right] \end{aligned}$$

The partition function for a single vortex in interaction with turbulence is

$$Z_V^{(0)}[J] = \mathcal{N}^{-1}\int D[\chi]\,D[\phi]\exp\left\{\int dxdy\left[\chi\left(\nabla_\perp^2\varphi-\alpha\varphi-\beta\varphi^2\right)+J\phi\right]\right\}$$

To calculate the generating functional of the vortex in the background turbulence we proceed in two steps: we solve the Euler-Lagrange equations obtaining the configuration of the system which extremizes this action; further, we expand the action to second order in the fluctuations around this extremum. This will include configurations that are close to the extremum of the action, *i.e.* turbulent waves and fluctuating shape of the vortex. The Euler-Lagrange equations have the solutions $\varphi_{Js}(x,y) \equiv \varphi_s(x,y)$ and $\chi_{Js}(x,y) = -\varphi_s(x,y) + \widetilde{\chi}_J(x,y)$. The first is the explicit form of the vortex. The dual function is $-\varphi_s(x,y)$ plus a term resulting from the excitation by J in its equation. This additional term $\widetilde{\chi}_J(x,y)$ is calculated by the perturbation of the KdV soliton solution according to the modification of the Inverse Scattering Transform when an inhomogeneous term (*i.e.* J) is included. The action functional is calculated for these two functions $S_{Vs}[J] \equiv S_V[\varphi_s(x,y), -\varphi_s(x,y) + \widetilde{\chi}_J(x,y)]$. Then the first part of our calculation is $Z_V^{(0)}[J] \sim \mathcal{N}^{-1} \exp\{S_{Vs}[J]\}$. Expanding the action around this extremum we calculate the Gaussian integral and obtain

$$Z_V^{(0)}[J] = \mathcal{N}^{-1} \exp\{S_{Vs}[J]\} \left[\det\left(\left.\frac{\delta^2 S_V[J]}{\delta\varphi\delta\chi}\right|_{\varphi_{Js},\chi_{Js}}\right)\right]^{-1/2}$$

The result is $Z_V^{(0)}[J] = \mathcal{N}^{-1} \exp\{S_{Vs}[J]\}\, A\, B$ where $A[J] \equiv \left[\frac{\beta/2}{\sinh(\beta/2)}\right]^{1/4}$, $B[J] \equiv \left[\frac{\sigma/2}{\sin(\sigma/2)}\right]^{1/2}$. The explicit expression for $\beta[J]$ and $\sigma[J]$ has been obtained in the references [1] and [2]. Finally, for the weakly interacting gas of vortices each of them affected by the interaction with random waves, we obtain

$$\begin{aligned}
\frac{1}{\phi_0^2}\langle\phi_V(x,y)\,\phi_V(x',y')\rangle_{\mathbf{k}} &= \frac{1}{k^2\rho_s^2}\left(1+\frac{1}{k^2\rho_s^2}\right)\frac{1}{8\pi^2}\left(1+\frac{\rho_s^2k^2+1}{\rho_s^2k^2+1-Z_V^{(0)}4\pi^2\rho_s^2/A}\right)\\
&+S(\mathbf{k})\,\frac{10\pi^2\rho_s^2}{A}\left\{\frac{1}{\rho_s^2k^2+1-Z_V^{(0)}4\pi^2\rho_s^2/A}+\frac{6}{5}\frac{\pi^2\rho_s^2}{A}\frac{1}{\left[\rho_s^2k^2+1-Z_V^{(0)}4\pi^2\rho_s^2/A\right]^2}\right\}\\
&+S(\mathbf{k})\,[1+f(\mathbf{k})]\left(1+\frac{2\pi^2\rho_s^2}{A}\frac{1}{\rho_s^2k^2+1-Z_V^{(0)}4\pi^2\rho_s^2/A}\right)
\end{aligned}$$

Here $S(\mathbf{k})$ is expressed in terms of Euler *beta* function and $f(\mathbf{k})$ is given in detail in [2].

For the turbulent, nonlinearly coupled waves scattered by vortices, our approach combines two distinct elements. The first is the inclusion of the vortices as a random perturbation in the generating functional of the turbulent waves. The second is a careful treatment of the intrinsic nonlinearity of the turbulent waves, already modified by the inclusion of the effect of the random vortices.

In the first part we follow the similar approaches as for the electron conduction in the presence of random impurities, or as for flexible polymers in porous media. For the case where the vortices have uncorrelated random positions and take at random positive or negative amplitudes of equal magnitude we find that the problem is mapped onto the *sine*-Gordon model (actually *sinh*). The functional integral is

$$\begin{aligned}
Z_J &= \mathcal{N}^{-1}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} ds\,\exp\left(-\frac{1}{2}s^2\right)\\
&\times\int D[\phi]\,D[\chi]\exp\left\{\int dxdy\left[-\chi\left(\nabla^2\phi\right)+\alpha\chi\phi+\beta\chi\phi^2-\sqrt{2Q}\chi\phi s+J\phi\right]\right\}
\end{aligned}$$

and can be done exactly. This leads to a renormalization of the coefficient which plays the role of a physical "mass" of the turbulent field, or, in other terms to a shift of the spatial scale from $\rho_s(1-v_d/u)^{-1/2}$ to higher values.

The second part consists of a systematic perturbative treatment of the nonlinearity in order to get, as much as possible, a correct representation of the nonlinear content of the field of the turbulent waves (this means the nonlinear interaction between low amplitude random waves). The nonlinearity is included in a functional perturbative treatment where the turbulent plasma is driven by random rise and decay of modes at marginal stability. We carry out a standard perturbative treatment for the differential equation $-\nabla^2\phi+\left(\alpha-s\sqrt{2A}\right)\phi+\beta\phi^2=\zeta$ with the objective to calculate correlation functions and other statistical properties. The noise is assumed with the simplest (white) statistics $\langle\zeta(x,y)\zeta(x',y')\rangle = D\delta(x-x')\delta(y-y')$ and should be considered as a random stirring force composed, for example, of random growths and decays of marginally stable modes, thus injecting at random places some energy into the system. After averaging over the random vortices it is possible to write $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_I$ where $\mathcal{L}_0=(\nabla\eta)(\nabla\phi)+\eta\alpha_s\phi+D\eta^2+J_\eta\eta+J_\phi\phi$ and $\mathcal{L}_I=\eta\beta\phi^2$. This induces a diffusive behavior of the turbulent field, at lowest order. We write

$$\begin{aligned} Z[\mathbf{J}] &= \int\mathcal{D}[\eta]\mathcal{D}[\phi]\exp\left(\int dxdy\mathcal{L}_0\right)\exp\left(\int dxdy\mathcal{L}_I\right) \\ &= \exp\left(\beta\int dxdy\frac{\delta}{\delta J_\eta}\frac{\delta}{\delta J_\phi}\frac{\delta}{\delta J_\phi}\right)\int\mathcal{D}[\eta]\mathcal{D}[\phi]\exp\left(\int dxdy\mathcal{L}_0\right) \end{aligned}$$

The vertex is $C_{ijk}(x,y)\equiv\beta\delta_{i\eta}\delta_{j\phi}\delta_{k\phi}$ and the propagator are $\widetilde{G}_{\phi\phi}=2D\left(k_x^2+k_y^2+\alpha_s\right)^{-2}$, $\widetilde{G}_{\phi\eta}=\widetilde{G}_{\eta\phi}=\left(k_x^2+k_y^2+\alpha_s\right)^{-1}$. We develop the treatment to one-loop, which means of order two in the strength of the nonlinearity, with the term $G_{\phi\eta}C_{\eta\phi\phi}G_{\phi\phi}G_{\phi\phi}C_{\phi\phi\eta}G_{\eta\phi}$. The result is

$$\langle\phi\phi\rangle_{\mathbf{k}}^{turbulence}=a\frac{2D}{(k^2+\alpha_s)^2}+b(2D\beta)^2\frac{1}{(k^2+\alpha_s)^2}\frac{1}{k^2(k^2+\alpha_s)^{3/2}} \quad (1)$$

where $\alpha_s\equiv\alpha-s\sqrt{2Q}$, $Q\equiv(2\beta\phi_0)^2/A$.

In conclusion, the various contributions to the spectrum provide $\mathbf{k}$ -power dependences which are in agreement with numerical simulations for plasma with intermittent behavior related to local organization in structures. It is necessary to quantify the relative weights of these contributions in connection with the experimental parameters characterizing these regimes.

References

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