THE MOST RECENT RESULTS IN MOMENT METHOD TO SOLVE THE BOLTZMANN EQUATION AND SETTING OF THE BOUNDARY PROBLEMS OF GAS AND PLASMA KINETIC THEORY

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1. Polynomial expansions

The non-linear moment method to solve the Boltzmann equation based on the distribution function presentation as a series in respect of the spherical Hermite polynomials is considered:

\[ f(v, r, t) = M(\alpha, c) \sum C_j(r, t) H_j(c), \]

\[ H_j = Y_{lm}(\theta, \varphi) c^j S_{l+1/2}^{(r)}(c^2), \quad c = \sqrt{\alpha}(v - u_0), \quad \alpha = \frac{m}{2kT_0}. \]

Here, \( M(\alpha, c) \) is a Maxwellian, \( Y_{lm}(\theta, \varphi) \) are the real spherical harmonics, \( S_{l+1/2}^{(r)}(c^2) \) are the Sonine (Laguerre) polynomials. An index \( j \) corresponds to three indices \( r, l, m \).

The relevant system of the moment equations takes a form as follows

\[ \frac{D(n_0(r, t) C_i)}{Dt} = \sum_{j,k} K^i_{j,k} C_j C_k, \]

\[ K^i_{j,k} = n_0^2 \int H_j \dot{I}(MH_j, MH_k) \, d^3v/g_i, \]

\( g_i \) is a normalizing coefficient, and \( \dot{I} \) is a Boltzmann collision operator. A differential operator \( D/ Dt \) refers to the left-hand parts of the moment equations (see, for example, [1]). One of the main obstacles in using this approach is a calculation of the matrix elements (MIEs) \( K^i_{j,k} \) corresponding to the moments of the non-linear collision integral. At the large indices, the direct formulas for ME calculation turn out to be extremely cumbersome with no chances to be handled even with the modern computers.

2. Recurrent relationships

Recently, we got the simple relationships between the MIEs [2]. These relationships are a corollary of the invariance property of the non-linear collision integral in respect of a choice of the basic functions and they are suit well for the recurrent procedure. For the axially symmetric case they take a form as follows

\[ (r_2 + 1)K^r_{l_1,l_2+1,l_2} = (\mu - R - L)K^r_{l_1,l_2+1,l_2} - rK^{r-1,l}_{l_1,l_2+1,l_2} - (r_1 + 1)K^r_{l_1+1,l_1+1,l_2}, \]

\[ R = r_1 + r_2 - r, \quad L = (l_1 + l_2 - l)/2, \]

\[ \beta(l_2)K^r_{l_1,l_2+1,l_2} = \beta(l - 1)K^r_{l_1,l_2+1,l_2} + \gamma(r - 1, l + 1)K^{r-1,l+1}_{l_1,l_2+1,l_2} - \gamma(r, l_1)K^r_{l_1+1,l_1+1,l_2} - \gamma(r_2, l_2)K^r_{l_1,l_1+1,l_2}. \]
Here,
\[ \beta(l) = -\frac{l + 1}{2l + 1}, \quad \gamma(r, l) = \frac{(r + 1)l}{2l + 1}. \]

A parameter \( \mu \) in Eq. (4) is determined through a dependence of the potential on a distance for the power potentials. With these recurrent relationships, it was proved a general Hecke theorem which gave a possibility to omit a number of the zero MEs and to reveal the additional relations between the isotropic linear MEs. According to this theorem, only those MEs can be non-zeros for which the following conditions would be valid
\[ |l_1 - l_2| \leq l \leq l_1 + l_2, \quad (-1)^l = (-1)^{l_1 + l_2}. \]

As a result, it was proved that any arbitrary ME concerning the non-oriented particles is a linear combination of the linear isotropic diagonal MEs. For the latters, the simple analytical formulas were built up.

Eqs. (4)–(5) are the corollaries of the invariance relative to a change in temperature and mean velocity of a weighted Maxwellian, relatively, in (1).

The relations between 3D matrix elements are the corollaries of the invariance relative to a turn in the coordinate system. It was shown that 3D matrix elements \( K_{r_1,l_1,m_1,r_2,l_2,m_2} \) are proportional to the relevant axially symmetric MEs \( K_{r_1,l_1,r_2,l_2} \), and the coefficients of proportionality are represented with the Klebsch-Gordan coefficients.

The universal codes were developed to ME calculation with the arbitrary indices at any cross section of interaction, i.e., for a Coulomb interaction between the particles. As a result, a perspective of easy and simple calculation of the MEs of the large indices can be advanced.

To verify the moment method’s feasibility, the calculations of several isotropic relaxation problems were carried out for a series of the interaction models [3]. It was shown that a distribution function can be built up to 10 times thermal velocity when involving a number of the moments (up to \( r_{max} = 128 \)).

Note, that, apart from the non-linear MEs, the linear non-isotropic MEs were obtained in a course of the calculations. In the standart kinetic theory, the linear matrix elements (brackets integral) are involved to calculate the transport coefficients. Commonly, they have been only calculated with no more than two-three Legendre polynomials. Even in this case, the calculations have been essentially tedious and the results have not been reliable. Applying the universal codes for the linear MEs is of a large progress in the domain of the large indices as well as in obtaining the reliable results. It has been revealed the errors in the present standard tables of the brackets integral [4].

When solving the Boltzmann equation with the moment method, there is a serious constraint involving a Grad criterion. To converge a expansion (1), it is necessary that a distribution function will be valid under a condition as follows
\[ \int_0^\infty f^2 \exp(e^2) d^3v < \infty. \]

E.g., this criterion for a shock wave is broken if the Mach number exceeds 1.85. The constraints related with the Grad criterion do not evolve on a stage of distribution function expansion in the spherical harmonics but on a stage of the expansion in the Sonine polynomials because these polynomials do be the orthogonal ones with a Maxwell weight. This constraint can be surmounted when expanding the distribution function in the spherical harmonics only.
3. Expansion of the collision integral in the spherically symmetric operators

Let an arbitrary distribution function is expanded in the spherical harmonics

\[ f(v) = \sum_{l,m} Y_{lm}^{l}(\theta, \varphi) \phi_{l,m}(v), \]

the expansion coefficient \( f_{l,m}(v) \) are dependent only on a velocity value. The coefficients \( C_{f} \) of a moment system (2) are expressed via the functions \( f_{l,m} \) as follows

\[ C_{f} = \frac{1}{\sigma_{r,l}} \int_{0}^{\infty} f_{l,m}(v) c f_{l,m} S_{l+1/2}^{r}(c^{2}) v^{2} dv. \]  

Here, \( \sigma_{r,l} \) are the normalizing coefficients of the Sonine polynomials. Let (7) to be substituted in a right-hand part of (2), and both parts to be multiplied by a function \( M(c)c^{l}S_{l+1/2}^{r}(c^{2}) \) then to be summed over \( r \). We obtain a system of the equations for \( f_{l,m} \) as follows

\[ \frac{Df_{l,m}(v)}{Dt} = \sum_{m_{1},m_{2},l_{1},l_{2}} \int_{0}^{\infty} \int_{0}^{\infty} G_{l_{1},m_{1},l_{2},m_{2}}^{l_{1},m_{1},l_{2},m_{2}}(v, v_{1}, v_{2}) f_{l_{1},m_{1}}(v_{1}) f_{l_{2},m_{2}}(v_{2}) v_{1}^{2} v_{2}^{2} dv_{1} dv_{2}. \]

Using the properties of the MEs, we have

\[ G_{l_{1},m_{1},l_{2},m_{2}}^{l_{1},m_{1},l_{2},m_{2}}(v, v_{1}, v_{2}) = \hat{Z}_{m_{1},m_{2}}^{m_{1},m_{2}}(l, l_{1}, l_{2}) G_{l_{1},l_{2}}^{l_{1},l_{2}}(v, v_{1}, v_{2}). \]

Here, the number coefficients \( \hat{Z}_{m_{1},m_{2}}^{m_{1},m_{2}}(l, l_{1}, l_{2}) \) are proportional to the Klebsch-Gordan coefficients, being independent on a cross section. The functions \( G_{l_{1},l_{2}}^{l_{1},l_{2}}(v, v_{1}, v_{2}) \) do not depend on the indices \( m_{1}, m_{2} \), being only dependent on the absolute values of velocities, i.e. are the kernels of the spherically symmetric operators. They are non-zeros only if \( l, l_{1}, l_{2} \) meet a general Hecke theorem (6) and can be built up through the axially symmetric MEs only:

\[ G_{l_{1},l_{2}}^{l_{1},l_{2}}(v, v_{1}, v_{2}) = M(\alpha, \alpha) \sum_{r} \sum_{r_{2}} \frac{1}{\sigma_{r_{1},l_{1}} \sigma_{r_{2},l_{2}}} \times \]

\[ S_{l+1/2}^{r}(c) K_{r_{1},l_{1},r_{2},l_{2}}^{r_{1}} S_{l_{1}+1/2}^{r_{1}}(c_{1}) S_{l_{2}+1/2}^{r_{2}}(c_{2}). \]

Thus, when knowing the non-linear MEs \( K_{r_{1},l_{1},r_{2},l_{2}}^{r_{1}} \), we can build up the kernels of the non-linear spherically symmetric operators for the arbitrary cross sections of interaction. Note, that, earlier, Hecke [5] obtained the simple analytical formulas for such kernels in a case of the hard-sphere model and linearized Boltzmann equation, using the Hilbert’s results [6]. Lately, in [7], the same kernels were built up. This made possible for the authors [7] to obtain an essential progress in solving a problem on a propagation of sound.

To finish with a statement of the problem on a level \( f_{l,m} \) it is necessary to have an expression concerning a differential part of an equation (8). E.g., for the axially symmetric case without any external force, it is expressed as follows

\[ \frac{Df_{l}(v, x, t)}{Dt} = \frac{\partial f_{l}(v, x, t)}{\partial t} + v \left( \frac{l}{2l - 1} \frac{\partial f_{l-1}(v, x, t)}{\partial x} + \frac{l + 1}{2l + 3} \frac{\partial f_{l+1}(v, x, t)}{\partial x} \right). \]
4. Formulation of the boundary conditions

With the kernels $G_{l_1,l_2}$ one can to be well in progress, also, when solving this problem. Indeed, a total collision integral can be expressed as follows (for brevity it is written for an axially symmetric case)

$$
\hat{I}(f, f) = \sum_l P_l(x) \sum_{l_1,l_2} \frac{(2l_1 + 1)(2l_2 + 1)}{4} \int_0^\infty \int_0^\infty G_{l_1,l_2}(v, v_1, v_2) \times \left( \int_{-1}^1 P_{l_1}(x_1)f(v_1)dx_1 \right) \left( \int_{-1}^1 P_{l_2}(x_2)f(v_2)dx_2 \right) v_1^2 v_2^2 dv_1 dv_2.
$$

Such approach in solving a number of the boundary problems was successfully realized in [8] for the linearized Boltzmann equation and the hard-sphere model when using the kernels from [5], [7].

Thus, beginning with a study on the MEs of the collision integral for the moment method, we ended with an essential simplification of the collision integral. The multiple integration over the complex areas in velocity space is omitted.

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References