A Non-Uniform Lattice Boltzmann Method for Fluid Flow Simulations

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Lattice Boltzmann Methods (LBM) can provide a mesoscopic description of the transport properties of physical systems using a BGK Boltzmann equation with a single time relaxation rule. The discretized form of LBM is solved using a split scheme collision step
\begin{equation}
\tilde{f}_a(x_i, t) = f_a(x_i, t) - \frac{1}{\tau} (f_a - f_a^{(eq)}),
\end{equation}
followed by simple advection of the post-collision state
\begin{equation}
 f_a(x_i + c_{a,i} \Delta t, t + \Delta t) = \tilde{f}_a(x_i, t),
\end{equation}
where $\tilde{f}_a$ is the post collision state, $\tau$ is the relaxation constant, with $f_a = f_a^{(eq)} + f_a^{(neq)}$. Since LBM algorithms are simple, inherently local and easily parallelized they are a powerful alternative to classical methods, which require the (computationally expensive) resolution of non-local nonlinear macroscopic convective derivatives.

Fine scale structures such as shocks, current sheets and vortex sheets arise in many plasma problems. Their resolution in any algorithm requires the introduction of non-uniform grids: coarse grids for large scales structures and fine grids for the small scale structures. The necessity of such approach for highly nonlinear MHD problems has been seen e.g. in our earlier LBM simulations of Orszag-Tang vortex \cite{1}.

While multi-grid algorithms are quite common in standard CFD, they have not received as much attention in LBM. Here, we present the use of multi-grids in LBM on the one dimensional (1D) Burgers equation,
\begin{equation}
\partial_t v(x) + v(x) \partial_x v(x) = \nu \partial_x^2 v(x).
\end{equation}
as a step toward application of the multi-grid technique to more complex flows. Burgers equation is an ideal test case since shocks develop in time.

The simplest LBM model for Burgers equation \cite{2} uses forward and backward streaming $c_{+,-}$ as well as a rest speed $c_0$. By enforcing the moments:
\begin{equation}
 u = \sum_a f_a^{(eq)}
\end{equation}
\begin{equation}
 \frac{1}{2} u^2 = \sum_a f_a^{(eq)} c_a
\end{equation}
\begin{equation}
 \frac{1}{3} u^3 + \lambda u = \sum_a f_a^{(eq)} c_a^2
\end{equation}
where $\lambda$ is an arbitrary constant that will appear in the viscosity term, the equilibrium distribution functions take the form
\begin{equation}
 f_0^{(eq)} = v - \frac{\lambda v}{c^2} - \frac{v^3}{3c^2},
\end{equation}
\begin{equation}
 f_\pm^{(eq)} = \frac{\lambda v}{2c^2} + \frac{v^2}{4c} + \frac{v^3}{6c^2},
\end{equation}
\end{equation}
\[ f(\text{eq}) = \frac{\lambda v}{2c^2} - \frac{v^2}{4c} + \frac{v^3}{6c^2}. \]  

The viscosity is then defined by \( \nu = \lambda (\tau - \frac{1}{2}) \).

An efficient method \([3, 4, 5]\) for non-uniformly refining the grid designates a refinement factor, which can easily be altered to preserve small gradients of the fields across regions containing strong shocks. The refinement factor is given by \( m \equiv \frac{\Delta x_c}{\Delta x_f} \) where \( \Delta x_c \) is the spatial step in the course grid and \( \Delta x_f \) is the spatial step in the fine grid. Preserving the coupling between the streaming vectors and the spatial lattice requires that the lattice vectors be defined as \( c_{a,i} \equiv \frac{\Delta x_c}{\Delta t_c} = \frac{\Delta x_f}{\Delta t_f} \).

This necessitates that the time also evolves on the fine grid in fractional increments of the course grid time steps. Thus we have a dual definition for the refinement factor \( m \equiv \frac{\Delta t_c}{\Delta t_f} \). This refined temporal and spatial stepping requires that the transfer of information between the course and fine grids be handled very carefully. In particular, we begin by defining the relaxation constant in the fine grid as a function of the relaxation constant in the course grid \( \tau_f = \frac{1}{2} + m(\tau_c - \frac{1}{2}) \) in order to preserve the same viscosity across the boundary between the fine and course grids. It is useful to include two overlapping points at each interface between the fine and course grids so that the fine distribution functions can be translated into course distribution functions and vice versa. The mathematical interface between the fine and course grids can be derived from Eqs. (1), (2), and the enforcement of uniformity of Eqs. (4), (5), and (6) across the interface boundary

\[ \tilde{f}_a^c = f_a^{(eq,f)} + m \frac{\tau_c - 1}{\tau_f - 1} (\tilde{f}_a^f - f_a^{(eq,f)}), \]  

\[ \tilde{f}_a^f = f_a^{(eq,c)} + \frac{1}{m} \frac{\tau_f - 1}{\tau_c - 1} (\tilde{f}_a^c - f_a^{(eq,c)}). \]

Figure 1 shows a flow chart of the computational procedure.

The key difference between uniform and non-uniform grid LBMs lies in the collision steps that occur at the boundary interfaces and in the fractional time stepping that occurs on the fine grid. As was explained previously, "m" time steps on the fine grid correspond to one time step on the coarse grid. A temporal interpolation is needed because the coarse grid does not contain streaming information for the fractional time steps, which are needed by the fine grid. For this interface point, we simply interpolate \( f_a^c(t + \frac{1}{m}) \) from \( f_a^c \) at \( t = t - 1 \), \( t = t \) and \( t = t + 1 \). This constitutes a second order Lagrange interpolation. During the first time step however, the distribution function at \( t = t - 1 \) is not available, so a simple average (or first order Lagrange interpolation) between the \( t = 0 \) and \( t = 1 \) is used.

The simplest non-trivial initial condition one can impose on the velocity profile consists of a sinusoidal wave that spans the entire periodic domain

\[ v(x, t_0) = \sin(2\pi \frac{x_i}{x_{\text{size}}}) \quad i = 0, 1, 2, \ldots, x_{\text{size}} - 1 \]  

where "xsize" is the total number of grid points and \( x_i \) is a discrete point positioned along the “x” axis. As the positive part of the Sin wave propagates to the right and the negative part propagates to the left a strong shock forms in the middle. This shock creates a large gradient in the velocity field.

We will test the benefits gained by this coupled lattice unmatched node LBM by comparing it against a standard LBM for the same number of total grid points. Figure 2 shows such a comparison for simulations containing 600 grid points.
Figure 1: A flow chart of the computational procedure of the non-uniform grid LBM.

Figure 2: A comparison of simulations of Burger’s equation using a uniform (blue) and coupled non-uniform (red) LBM, for 600 grid points and Re=3600. The velocity profiles are plotted for t=0, t=2, and t=4 dimensionless time steps. No numerical oscillations are visible in the non-uniform grid simulation.

The standard LBM uniformly distributes the points over the periodic regime while the coupled lattice unmatched procedure places 300 grid points in the inner 1/5 of the plot and spreads the remaining 300 grid points over the outer regions. The refinement factor was thus set to $m = 4$ to accommodate this distribution. 600 LBM time steps on the uniform
grid correspond to 375 LBM time steps on the non-uniform grid and both correspond to one dimensionless time step. Both simulations were run for 4 dimensionless time steps with an initial velocity of $v_0 = 1$. For the uniform grid simulation, the relaxation constant was set to $\tau = 0.52$ and the free parameter was set to $\lambda = \frac{25}{3}$. This gives a viscosity of $\nu = 0.1667$, and a dimensionless Reynolds number of $Re = 3600$. For the non-uniform grid simulation, the relaxation constant was set to $\tau = 0.5125$ and the free parameter was set to $\lambda = \frac{25}{7}$. This gives a viscosity of $\nu = 0.1042$, and a dimensionless Reynolds number of $Re = 3600$. The field profile in dimensionless time steps is plotted for $t=0$, $t=2$, and $t=4$. The non-uniform grid LBM velocity profiles do not contain the numerical oscillations that are clearly visible in the standard LBM simulation.

The total energy given by

$$ E(t) = \int_0^{2\pi} \frac{1}{2} v(x, t)^2 dx $$

is plotted in Fig. 3. The energy is initially conserved during the initial evolution of the profile. As the velocity shocks collide, the energy begins to dissipate due to the presence of the viscosity term.

![Figure 3: Total energy of the non-uniform grid LBM versus time.](image)

This non-uniform grid LBM produces a robust methodology for simulating fluid flows. It allows us to refine the grid in the regions where large gradients in the fields develop thus decreasing the field gradients with respect to the grid density. This process achieves higher numerical accuracy resulting in the elimination of numerical oscillations around shock fronts.

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**References**