Derivation of a Reynolds stress response functional for zonal flows from numerical simulations

N. Guertler, K. Hallatschek
Max-Planck-Institut für Plasmaphysik, Garching, Germany

Introduction

Self-consistent numerical studies of ITG-turbulence, using the NLET two fluid code [1], show a Reynolds stress driven zonal flow pattern with a characteristic radial scale length. Based on these observations the requirements for a response functional that can predict the time-evolution and scales of the zonal flows will be analyzed. The long term goal of this research in excitation, damping and time-evolution of zonal flows is to achieve a better understanding of the high-confinement mode and internal transport barriers in fusion devices.

Characteristic scale

The NLET code was used to simulate the turbulent ITG-system with circular geometry using reduced Braginskii equations with adiabatic electrons described in [1],[2]. The parallel ion heat conductivity was chosen such that temperature perturbations achieve similar damping rates as in kinetic theory. The fluid code is computationally more practical for this analysis than a gyro-kinetic code since a large number of high resolution, long time simulations are required.

Simulations of the core where zonal flows are prevalent show a characteristic flow pattern, fig. 1. The upper figure shows the color coded time-evolution of the flux surface average zonal flow velocity. The flows repel each other and, since the boundary conditions allow them to drift out of the computational domain, the radial spectrum may change slowly over time, lower figure. However, the radial...
scale length only varies within tight boundaries as new flows grow to maintain a characteristic scale. Simulations starting from modified initial states with artificial flow patterns always decay into the characteristic zonal flow pattern indicating the robustness of the scale and deterministic flow evolution.

**Derivation of a stress response functional**

Based on wave-kinetic theory, a response functional for the Reynolds stress $R$ can be constructed, eq. (2), attempting to describe the excitation, saturation and time-evolution of the flux surface average zonal flow velocity $\overline{v}_\theta$ using the poloidal force balance, eq. (1) [3]. $R$ contains both the perpendicular and parallel stress components [2]. $\alpha, \beta, \gamma$ are constants and $I$ the turbulence intensity, e.g. radial heat transport.

\begin{align*}
\partial_t \overline{v}_\theta &= -\partial_r R \\
R &= I \left[ \alpha \partial_r \overline{v}_\theta \left( 1 - \beta (\partial_r \overline{v}_\theta)^2 \right) + \gamma \partial_r^3 \overline{v}_\theta \right] \tag{2}
\end{align*}

For small shearing rates $u := \partial_r \overline{v}_\theta$ only the first term $\alpha \partial_r \overline{v}_\theta$ is relevant and results, inserted in eq. (1), in a zonal flow growth. The non-linear second term becomes more dominant as the shearing rate increases and leads to a flow saturation when $\beta (\partial_r \overline{v}_\theta)^3 \gtrsim \partial_r \overline{v}_\theta$. The third $\gamma \partial_r^3 \overline{v}_\theta$ term models the observed finite width of the flow peaks by damping modes with high $k_r$. The turbulence intensity $I$ in (2) accounts for the proportionality of the stress and the local turbulence level.

Insertion of a self-consistent time-evolution of a flow pattern into the functional (2) reproduces the associated stress pattern quite well, using appropriate values for $\alpha, \beta$ and $\gamma$. However, the prediction of the observed flow pattern from an initial flow state fails, as a numerical simulation of (1) and (2) always shows the same behavior, fig. 2: Modes with high $k_r$ are quickly damped and flows with low $k_r$ decay into the largest mode that fits into the domain. This is contrary to the turbulence simulations where arbitrary initial flow states always decay into the characteristic zonal flow pattern.

A mean-field theory approximation of $\partial_r \langle u \rangle^3 \approx 3 \langle u^2 \rangle k_r^2 \overline{v}_\theta$ yields an estimate for the zonal flow growth rate $\Gamma$, given by eq. (3). The average is denoted by $\langle \rangle$. The $k_r^4$ term damps large $k_r$ for all shearing rates $u$. An increase in the shearing rate $u$ further confines the $k_r$ region with positive growth rate from the high $k_r$ side. When $\beta \langle u^2 \rangle$ approaches $1/3$ only a small region of growing $k_r$ around $k_r = 0$ remains, explaining the decay of the flow pattern into the smallest possible $k_r$.

\[ \Gamma(k_r, u) = I k_r^2 \left( \alpha (1 - 3 \beta \langle u^2 \rangle) - \gamma k_r^2 \right) \tag{3} \]

Observations of quick damping of small $k_r$ in self-consistent simulations suggest an extension to
the response functional that confines the region with positive growth rate from the small \( k_r \) side. A possible behavior for the growth rate that would eliminate the decay into the largest mode is sketched in figure 4. For a non-zero shearing rate \( u \) the small \( k_r \) modes would be damped before the higher modes. This would leave a localized region of intermediate \( k_r \) where flows continue to grow. This region would get more confined until all \( k_r \) are damped and the flows saturate at a distinct zonal flow scale.

One way to obtain this dip for small \( k_r \) that respects the symmetry requirements of the fluid equations, is a change of sign in the third term to allow a damping of small \( k_r \) through the first and second term while the intermediate \( k_r \) still have a positive growth rate. The damping at the high \( k_r \) end is handled by a new forth term, eq. (4), with a constant factor \( \delta \).

\[
\Gamma(k_r, u) = I k_r^2 \left( \alpha(1 - 3\beta \left\langle u^2 \right\rangle) + \gamma (k_r^2 - \delta k_r^4) \right)
\] (4)

Translation of this growth rate behavior into a response functional yields equation (5). Numerical solution of (1) with this new stress functional, starting from arbitrary initial states, now results in stationary flow pattern with an characteristic intrinsic scale, fig. 3, depending on \( \gamma \) and \( \delta \).

\[
R = I \left[ \alpha \partial_r \nu_\theta \left( 1 - \beta (\partial_r \nu_\theta)^2 \right) - \gamma \left( \partial_r^3 \nu_\theta + \delta \partial_r^5 \nu_\theta \right) \right]
\] (5)
To verify the extended functional, turbulence simulations with artificially maintained sinusoidal zonal flow patterns of different $k_r$ and shearing rates were used. Turbulence effects were removed by averaging over long time scales and large domains to obtain the deterministic stress response to the flow pattern. The resulting growth rate estimates are shown in figure 5. For a low shearing rate of $u = 0.1$ the growth rates for all modes are still positive. At a higher shearing rate of $u = 0.2$ the expected dip in the growth rate for low $k_r$ occurs while a region of intermediate modes still grows. This corroborates the proposed behavior sketched in figure 4 and is in accordance with the observations in self-consistent simulations.

Conclusions

It was demonstrated that the Reynolds stress response functional (2), which already quantitatively reproduces the stress patterns of ITG-turbulence zonal flow simulations, requires an extension (5) to predict the time-evolution and characteristic intrinsic scale of the zonal flows. Artificial flows where used in turbulence computations to verify the damping at small $k_r$ while an intermediate range of modes is still growing. Additional analysis of the high $k_r$ behavior, the nonlinear interaction and dependence on the turbulence intensity gradient $\partial_r \ln I$ is required for a complete verification of the new response functional.

References