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TRANSPORT EQUATIONS FOR FAST IONS IN TURBULENT PLASMA

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A theoretical method for describing the fast-ion transport in a presence of electric and magnetic fluctuations is presented. In this paper we concentrate only on the beam-induced parallel current, although this method can be applicable to the number density and the energy density of fast ions.

Let us write a gyrophase-averaged distribution function for fast ions as $f(\boldsymbol{r},\boldsymbol{v},t)$ and start with a Fokker-Planck equation for this distribution function in a presence of fluctuations:

$$\partial_t f + v_\parallel \boldsymbol{b}\cdot\nabla f + \delta\boldsymbol{v}_\perp\cdot\nabla_\perp f - C(f) = \frac{\dot{n}_b}{2\pi v^2}\delta(v-v_b)\Theta(\zeta) \equiv S(\boldsymbol{r},\boldsymbol{v}), \qquad (1)$$

where the subscripts $\parallel$ and $\perp$ refer to the parallel and the perpendicular to a constant averaged-magnetic-field $\boldsymbol{B}$; $\boldsymbol{b} = \boldsymbol{B}/B$; $v_\parallel = \boldsymbol{v}\cdot\boldsymbol{b}$; $\zeta = v_\parallel/|v_\parallel|$; $\boldsymbol{v} = (v,\zeta)$; $\delta\boldsymbol{v}_\perp$ is the velocity fluctuation ;and S represents a monoenergetic fast-ion source term. We write the velocity fluctuations as $\delta\boldsymbol{v}_\perp = \boldsymbol{b}\times\nabla\delta\phi(\boldsymbol{r},t)/B$ for electrostatic fluctuations and $\delta_\perp\boldsymbol{v} = v_\parallel\delta\boldsymbol{B}_\perp/B \equiv v_\parallel\delta\boldsymbol{b}_\perp(\boldsymbol{r},t)$ for magnetic fluctuations, where $\delta\phi$ is the fluctuating electrostatic potential and $\delta\boldsymbol{B}_\perp$ is the fluctuating perpendicular magnetic field. The Coulomb collision operator C is approximated by the slowing-down term and the pitch-angle-scattering term:

$$C(f) = \frac{1}{\tau_s}\frac{1}{v^2}\frac{\partial}{\partial v}\left[(v^3+v_c^3)f\right] + \frac{\hat{Z}}{2\tau_s}\frac{v_c^3}{v^3}\frac{\partial}{\partial\zeta}(1-\zeta^2)\frac{\partial f}{\partial\zeta} \qquad (2)$$

with the critical velocity v_c, the slowing-down time τ_s and $\hat{Z} = \sum_i n_i e_i^2 \ln\Lambda_i / \left(\sum_i n_i e_i^2 m_b \ln\Lambda_i/m_i\right)$, where the subscripts i and b denote the bulk and fast ions, respectively.

A closed set of equations for an ensemble-averaged distribution function and a response function to an infinitesimal external perturbation is derived by applying a renormalized perturbation theory to the Fokker-Planck equation (1) [1]. In the steady state the equation for the ensemble-averaged distribution function $\bar{f}(\boldsymbol{r},\boldsymbol{v}) = \langle\langle f(\boldsymbol{r},\boldsymbol{v},t)\rangle\rangle$ becomes

$$(v_\parallel\boldsymbol{b}\cdot\nabla - C)\bar{f}(\boldsymbol{r},\boldsymbol{v}) - \nabla_\perp\cdot\int d\boldsymbol{v}'\,\boldsymbol{\mathcal{D}}(\boldsymbol{v}',\boldsymbol{v})\cdot\nabla_\perp\bar{f}(\boldsymbol{r},\boldsymbol{v}') = S(\boldsymbol{r},\boldsymbol{v}) \qquad (3)$$

with $\boldsymbol{\mathcal{D}}(\boldsymbol{v}',\boldsymbol{v}) = \int_0^\infty d\tau\int d\boldsymbol{r}'\,\boldsymbol{F}(\boldsymbol{r}',\tau,\boldsymbol{v},\boldsymbol{v}')G^\dagger(\boldsymbol{r}',-\tau,\boldsymbol{v}';\boldsymbol{v})$, where $\langle\langle\ \cdot\ \rangle\rangle$ denotes the ensemble average over fluctuations, $G^\dagger(\boldsymbol{r}',-\tau,\boldsymbol{v}';\boldsymbol{v})$ is the response function, $\boldsymbol{F}(\boldsymbol{r}-\boldsymbol{r}',t-t',\boldsymbol{v},\boldsymbol{v}') = \langle\langle\delta\boldsymbol{v}_\perp(\boldsymbol{r},\boldsymbol{v},t)\delta\boldsymbol{v}_\perp(\boldsymbol{r}',\boldsymbol{v}',t')\rangle\rangle$ is the correlation function of fluctuations.

Next, averaging the equation (1) over y and z and denoting this average by $\langle \ \cdot \ \rangle$, we obtain

$$-C(\bar{f}(x,\boldsymbol{v})) - \frac{\partial^2}{\partial x^2}\int d\boldsymbol{v}' D_x(\boldsymbol{v}',\boldsymbol{v})\bar{f}(x,\boldsymbol{v}') = S(x,\boldsymbol{v}), \tag{4}$$

where $\bar{f}(x,\boldsymbol{v}) = \langle \bar{f}(\boldsymbol{r},\boldsymbol{v})\rangle$, $S(x,\boldsymbol{v}) = \langle S(\boldsymbol{r},\boldsymbol{v})\rangle$ and $D_x(\boldsymbol{v}',\boldsymbol{v}) = \boldsymbol{e}_x\cdot\boldsymbol{\mathcal{D}}(\boldsymbol{v}',\boldsymbol{v})\cdot\boldsymbol{e}_x$. Let us now introduce the function $\varphi(\boldsymbol{v}) = e_b\,\zeta\,[(v^3+v_c^3)/v^3]^{\hat{Z}/3}\int_0^v dv'[v'^3/(v'^3+v_c^3)]^{\hat{Z}/3+1} \equiv e_b\,\zeta\,\hat{\varphi}(v)$, which is the solution to the equation $C^\dagger(\varphi(\boldsymbol{v})) = -e_b v_\parallel/\tau_s$. Using this function and (4), we express the parallel current density $J_b(x)$ of fast ions in the form:

$$J_b(x) = J_{b0}(x) + \tau_s\frac{d^2}{dx^2}\int d\boldsymbol{v}\, A(\boldsymbol{v})\bar{f}(x,\boldsymbol{v}), \tag{5}$$

where $A(\boldsymbol{v}) = \int d\boldsymbol{v}'\varphi(\boldsymbol{v}')D_x(\boldsymbol{v},\boldsymbol{v}')$ and $J_{b0}(x) = \tau_s\int d\boldsymbol{v}\,\varphi(\boldsymbol{v})S(x,\boldsymbol{v})$ is the parallel current density in the absence of fluctuations. Furthermore, introducing the function $\chi(\boldsymbol{v})$ satisfying the equation $C^\dagger(\chi(\boldsymbol{v})) = A(\boldsymbol{v})$ and denoting the relation $\int d\boldsymbol{v}\,A(\boldsymbol{v})\bar{f}(x,\boldsymbol{v}) = \int d\boldsymbol{v}\,\chi(\boldsymbol{v})C(\bar{f}(x,\boldsymbol{v})) \simeq -\int d\boldsymbol{v}\,\chi(\boldsymbol{v})S(x,\boldsymbol{v})$, we present the following diffusion equation for the parallel current density:

$$\tau_s D\,\frac{d^2}{dx^2}\,J_b(x) - J_b(x) = -J_{b0}(x), \tag{6}$$

where the diffusion coefficient is written as

$$D \simeq -\frac{\int d\boldsymbol{v}\,\chi(\boldsymbol{v})S(x,\boldsymbol{v})}{J_{b0}(x)} = -\frac{\int d\boldsymbol{v}\,\chi(\boldsymbol{v})S(x,\boldsymbol{v})}{\tau_s\int d\boldsymbol{v}\,\varphi(\boldsymbol{v})S(x,\boldsymbol{v})}. \tag{7}$$

We next explicitly calculate the diffusion coefficient by assuming the correlation functions of fluctuating electrostatic potential and fluctuating magnetic field in the form: $\langle\langle\delta\phi(\boldsymbol{r},t)\delta\phi(\boldsymbol{0},0)\rangle\rangle = \phi_0^2\hat{F}(\boldsymbol{r},t)$ and $\langle\langle\delta\boldsymbol{b}_\perp(\boldsymbol{r},t)\delta\boldsymbol{b}_\perp(\boldsymbol{0},0)\rangle\rangle = -\beta^2\lambda_x\lambda_y(\nabla\times\boldsymbol{e}_z)(\nabla\times\boldsymbol{e}_z)$ $\hat{F}(\boldsymbol{r},t)$ with $\hat{F}(\boldsymbol{r},t) = \exp\left[-|t|/\tau_c - x^2/2\lambda_x^2 - y^2/2\lambda_y^2 - z^2/2\lambda_\parallel^2\right]$.

For electrostatic fluctuations, the equation for the response function is approximately solved in two regimes. The regime in which the diffusion term in this equation can be neglected is referred to as the weak turbulent regime. When the collision term can be neglected, we refer to as the strong turbulent regime. Then the diffusion coefficients in the weak and strong turbulent regimes are obtained in the form:

$$D = \frac{\xi_c\lambda_x^2}{\tau_s}\,\frac{\int_{-1}^1 d\zeta\,\zeta\Theta(\zeta)\,D(v_b,\zeta)}{\hat{\varphi}(v_b)\int_{-1}^1 d\zeta\,\zeta\Theta(\zeta)} \tag{8}$$

with

$$D(v_b,\zeta) = \xi_c\int_0^{v_b} dv'\frac{v'^2}{v'^3+v_c^3}\int_0^{v'} d\bar{v}\frac{\bar{v}^2}{\bar{v}^3+v_c^3}\left(\frac{\bar{v}^3+v_c^3}{v'^3+v_c^3}\right)^{\frac{1}{3}\frac{\tau_s}{\tau_c}}$$

$$\times R(v_b,\bar{v})\hat{\varphi}(\bar{v})\exp\left\{-\frac{1}{2}\left[\frac{\tau_s}{\lambda_\parallel}\zeta R(v_b,v')V(v',\bar{v})\right]^2\right\} \tag{9}$$

for the weak turbulent regime, and

$$D(v_b,\zeta)=\frac{1}{\sqrt{2}}\int_0^{v_b}dv'\frac{v'^2}{v'^3+v_c^3}D(\tau_{ce}^{-1};a')R(v_b,\bar{v})\hat{\varphi}(v') \tag{10}$$

for the strong turbulent regime, where $\xi_c=\bar{\phi}_0\tau_s/\lambda_x\lambda_y$ with $\bar{\phi}_0=\phi_0/B$; $R(v,\bar{v})=\left[(v^3+v_c^3)\bar{v}^3/v^3(\bar{v}^3+v_c^3)\right]^{\hat{Z}/3}$; $V(v,v')=\int_{v'}^{v}d\bar{v}\bar{v}^3R(v,\bar{v})/(\bar{v}^3+v_c^3)$; $a'=[(v_b^3+v_c^3)v'^3/(v'^3+v_c^3)v_b^3]^{\hat{Z}/3}a$ with $a=\lambda_x\lambda_y v_b|\zeta|/\sqrt{2}\bar{\phi}_0\lambda_\parallel$; and $D(\tau_{ce}^{-1};a')=D(\tau_{ce}^{-1})/[1+\sqrt{2/\pi}a'D(\tau_{ce}^{-1})]$ with $D(\tau_{ce}^{-1})=\tau_{ce}/(1+\tau_{ce})$ and $\tau_{ce}=\sqrt{2}\bar{\phi}_0\tau_c/\lambda_x\lambda_y$.

Similarly the diffusion coefficient due to the magnetic fluctuations in the collisionless regime is obtained

$$D=\frac{\tau_{mb}\lambda_x^2}{\tau_c}\frac{\int_{-1}^{1}d\zeta\,\zeta\,|\zeta|\,\Theta(\zeta)\,D(v_b,\zeta)}{\hat{\varphi}(v_b)\int_{-1}^{1}d\zeta\,\zeta\Theta(\zeta)} \tag{11}$$

with

$$D(v_b,\zeta)=\frac{1}{2}D(\tau)\int_0^{v_b}dv'\frac{v'^2}{v'^3+v_c^3}R(v_b,v')^2\hat{\varphi}(v'), \tag{12}$$

where $\tau_{mb}=\sqrt{2}v_b\tau_c/\lambda_\parallel$; and $D(\tau)=\alpha D_0/(\alpha+D_0)$ with $\tau=\tau_{mb}|\zeta|$, $\alpha=\beta\lambda_\parallel/\sqrt{\lambda_x\lambda_y}$ and $D_0=\alpha^2\sqrt{\pi}e^{\tau^{-2}}\,\mathrm{erfc}\,(\tau^{-1})$.

The parallel momentum transfer from fast ions to electrons in collisions induces an electron return-current. Thus the resulting current density driven by fast ions is the sum of the current density $J_b(x)$ and the electron return-current density $J_e(x)$. When the effect of fluctuations on the return current is neglected, we have $J_e(x)\simeq-(Z_b/Z)J_b(x)$ in the uniform magnetic field , where Z_b is the charge state for the fast ion, Z is the effective charge for the bulk ions. In considering the fluctuation effect, we can derive the diffusion equation similar to (6) for the return current.

The scalings of the diffusion coefficient in the various asymptotic regimes are shown in tables 1 and 2. These asymptotic diffusion coefficients can be understood by a random walk picture: $D\sim(\delta v_\perp\tau_d)^2/\tau_d=(\delta v_\perp)^2\tau_d$, where $\delta v_\perp=\bar{\phi}_0/\lambda_x$ for electrostatic fluctuations; $\delta v_\perp=(\delta B_\perp/B)v_\parallel=(\lambda_x\alpha/\lambda_\parallel)v_\parallel$ for magnetic fluctuations; and the decorrelation time τ_d is the shortest time among τ_c, τ_s, λ_x^2/D and $\lambda_\parallel/|v_{\parallel b}|$ with $v_{\parallel b}=v_b\zeta_b$. The decorrelation times in the asymptotic regimes are also shown in tables 1 and 2.

Up to here, our formulation is based upon the closed set of equations for the ensemble-averaged distribution function and the response function that is derived by using the

Table 1: Asymptotic forms of the diffusion coefficient D for electrostatic fluctuations in the weak and strong turbulent regimes.

		τ_d	D(weak)			τ_d	D (strong)
$\frac{\lvert v_{\parallel b}\rvert \tau_s}{\lambda_\parallel} \ll 1$	$\tau_c \ll \tau_s$	τ_c	$\frac{\tau_c \xi_c^2 \lambda_x^2}{\tau_s^2}$	$a \ll 1$	$\tau_{ce} \ll 1$	τ_c	$\frac{\tau_c \xi_c^2 \lambda_x^2}{\tau_s^2}$
	$\tau_c \gg \tau_s$	τ_s	$\frac{\xi_c^2 \lambda_x^2}{\tau_s}$		$\tau_{ce} \gg 1$	$\frac{\lambda_x}{\delta v_\perp}$	$\frac{\xi_c \lambda_x^2}{\tau_s}$
$\frac{\lvert v_{\parallel b}\rvert \tau_s}{\lambda_\parallel} \gg 1$	$\frac{\lvert v_{\parallel b}\rvert \tau_c}{\lambda_\parallel} \ll 1$	τ_c	$\frac{\tau_c \xi_c^2 \lambda_x^2}{\tau_s^2}$	$a \gg 1$	$a\tau_{ce} \ll 1$	τ_c	$\frac{\tau_c \xi_c^2 \lambda_x^2}{\tau_s^2}$
	$\frac{\lvert v_{\parallel b}\rvert \tau_c}{\lambda_\parallel} \gg 1$	$\frac{\lambda_\parallel}{\lvert v_{\parallel b}\rvert}$	$\frac{\xi_c^2 \lambda_\parallel \lambda_x^2}{\lvert v_{\parallel b}\rvert \tau_s^2}$		$a\tau_{ce} \gg 1$	$\frac{\lambda_\parallel}{\lvert v_{\parallel b}\rvert}$	$\frac{\xi_c^2 \lambda_\parallel \lambda_x^2}{\lvert v_{\parallel b}\rvert \tau_s^2}$

Table 2: Asymptotic forms of the diffusion coefficient D for magnetic fluctuations.

		τ_d	D
$\alpha \ll 1$	$\tau_{mb}\lvert\zeta_b\rvert \ll 1$	τ_c	$\alpha^2 v_{\parallel b}^2 \lambda_x^2 \tau_c / \lambda_\parallel^2$
	$\tau_{mb}\lvert\zeta_b\rvert \gg 1$	$\lambda_\parallel / \lvert v_{\parallel b}\rvert$	$\alpha^2 \lambda_x^2 \lvert v_{\parallel b}\rvert / \lambda_\parallel$
$\alpha \gg 1$	$\alpha\tau_{mb}\lvert\zeta_b\rvert \ll 1$	τ_c	$\alpha^2 v_{\parallel b}^2 \lambda_x^2 \tau_c / \lambda_\parallel^2$
	$\alpha\tau_{mb}\lvert\zeta_b\rvert \gg 1$	$\lambda_x / \delta v_\perp$	$\alpha \lambda_x^2 \lvert v_{\parallel b}\rvert / \lambda_\parallel$

spatially local approximation. We finally discuss the spatially nonlocal effect on the transport equation for fast ions.

Without using the spatially local approximation, the equation for $\bar{f}(x, \boldsymbol{v})$ is obtained in the form:

$$-C(\bar{f}(x, \boldsymbol{v})) - \frac{\partial^2}{\partial x^2} \int d\boldsymbol{v}' \int_{-\infty}^{\infty} dx' K(x - x', \boldsymbol{v}', \boldsymbol{v}) \bar{f}(x', \boldsymbol{v}') = S(x, \boldsymbol{v}), \tag{13}$$

where $K(x - x', \boldsymbol{v}', \boldsymbol{v}) = \int_0^\infty d\tau \int dy\, dz\, F_x(\boldsymbol{r} - \boldsymbol{r}', \tau, \boldsymbol{v}, \boldsymbol{v}') G^\dagger(\boldsymbol{r} - \boldsymbol{r}', -\tau, \boldsymbol{v}'; \boldsymbol{v})$. We here approximately calculate the function K using the response function obtained in the spatially local approximation, and furthermore we concentrate only on the parameter regions $a \ll 1$ and $\tau_{ce} \gg 1$ for electrostatic fluctuations; and $\alpha \gg 1$ and $\alpha\tau_{mb}|\zeta_b| \gg 1$ for magnetic fluctuations, where the spatially nonlocal effect is most important. Then repeating the procedure leading to (6) yields the spatially nonlocal transport equation for the parallel current density:

$$\tau_s \frac{d^2}{dx^2} \int_{-\infty}^{\infty} dx' \; \tilde{D}(x - x')\, J_b(x') - J_b(x) = -J_{b0}(x), \tag{14}$$

where $\tilde{D}(x - x') = D\, \bar{K}(x - x')/\lambda_x$ with

$$\bar{K}(x - x') = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\tau \, \frac{1}{(1+\tau)^{3/2}\sqrt{\tau}} \exp\left[-\frac{1+\tau}{2\tau} \frac{(x - x')^2}{\lambda_x^2} \right]. \tag{15}$$

Reference

[1] M. Taguchi, Proc. 35th EPS Plasma Physics Conf. (2008) P4.183, Crete.