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SIMULATION OF CHARGED PARTICLE MOTION IN A HOMOGENEOUS MAGNETIC FIELD SUBJECT TO A RANDOM FORCE

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Following Langevin, the system of equations describing the motion of a charged particle in a uniform, homogeneous, stationary magnetic field **B**, subject to additive random forces modelling the interaction with the plasma, is

$$\frac{d\boldsymbol{r}(t)}{dt} = \boldsymbol{u}(t) \qquad \text{and} \qquad \frac{d\boldsymbol{u}(t)}{dt} = \Omega\left[\boldsymbol{u}(t)\times\boldsymbol{e_x}\right] - \gamma\boldsymbol{u}(t) + c\boldsymbol{\xi}(t).$$

Here, $\boldsymbol{r}(t)$ is the position and $\boldsymbol{u}(t)$ the velocity of the particle, while $\boldsymbol{e_x}$ is the unit vector in the x-direction. The parameters $\Omega[= q\,|\mathbf{B}|\,/m]$, γ and c represent the cyclotron frequency, the friction coefficient, and the strength of the random component of the force, respectively. It is assumed that this force consists of a deterministic component proportional to the velocity, and a fluctuating component $\boldsymbol{\xi}(t)$, usually referred to as "noise". In this work the stochastic term is assumed to be a Gaussian stationary process with vanishing mean and (diagonal) covariance matrix $\left\langle \xi_i(t)\,\xi_i(t+\tau)\right\rangle = \delta_{ij}\,\phi(|\tau|)$. Simulations have been performed with the following four correlation functions $\phi(|\tau|)$:

i) White noise: $\phi(|\tau|) = \delta(\tau)$ ii) Gaussian noise: $\phi(|\tau|) = \left(\varepsilon\sqrt{\pi}\right)^{-1}\exp\left(-\tau^2/\varepsilon^2\right)$

iii) Coloured noise: $\phi(|\tau|) = (2\varepsilon)^{-1}\exp\left(-|\tau|/\varepsilon\right)$ iv) Lorentzian noise: $\phi(|\tau|) = \varepsilon\left[\pi\left(\tau^2+\varepsilon^2\right)\right]^{-1}$

Note that all noises approach the white noise as $\varepsilon \to 0$, while they vanish as $\varepsilon \to \infty$.

Clearly, this problem leads to two independent sets of equations: an equation for the (one-dimensional) longitudinal motion along the direction of the field and a system for the (two-dimensional) transverse motion perpendicular to the field. Because of the linearity of the equations, formal solutions can be written in a straightforward manner and expressions for the expectation values of powers of the position and/or the velocity of the particle are obtained for white noise (see e.g. [1]) and, in fact, for all choices of $\phi(|\tau|)$. Numerical simulations are considered here and analytical solutions are used to test their accuracy.

For Brownian motion, i.e. white noise, numerical simulations have been carried out [2], approximating the random acceleration $\xi(t)\mathrm{d}t$ by $\sqrt{\Delta t}\,\mathrm{N}(t)$, where N($t$) is a normal random variable with vanishing mean and unit variance. This procedure can also be extended to the case of coloured noise, but it cannot be applied to other correlations, such as the Gaussian or the Lorentzian. Such correlations, which are of some interest when random electric and/or magnetic fields are considered, may be treated by Fourier methods [3] and then, following Billah and Shinozuka [4], the stochastic acceleration is approximated by the Fourier series

$$\xi(t) = \sqrt{2}\sum_{n=1}^{N}\sqrt{2S_{\xi}(\omega_n)\Delta\omega}\cos(\omega_n t + \Phi_n).$$

The above approximation depends on the spectral energy density (SED) of the noise $S_{\xi}(\omega_n)$, i.e. the Fourier transform of $\phi(|\tau|)$, and a set of statistically independent and uniformly distributed in $[0, 2\pi]$ random angles Φ_n. The frequency is discretized in steps of length $\Delta\omega$, depending on the number N of terms, according to $\omega_n = n\Delta\omega$ and $\Delta\omega = \omega_{\max}/N$, where $\omega_{\max}$ the maximum frequency considered. This approach can be applied to any type of non-white noise by substituting the appropriate SED. Other approaches simulating coloured noise require an explicit equation for the time evolution of the force. The result is an ergodic, Gaussian process and a substantial portion of the "tail" is reproduced, a characteristic which is missing in other methods employing the Box-Mueller algorithm. The time step is also larger than in other methods.

An extensive parametric study has been performed for a wide range of various physical and computational parameters, especially the correlation time and the friction coefficient. The values of γ have been selected to be low because we are interested in the behaviour of the motion of charged particles in low-friction mediums, like fusion plasma. The numerical results have been compared with the analytical solutions as this investigation serves for validating and benchmarking the implemented computational scheme.

The average and variance of position and velocity of the particle are calculated in every time step and a representative case of motion along the magnetic field is shown in Figure 1. Both analytical and numerical curves are included for the case of coloured noise and the agreement is very good, thus establishing confidence in the numerical method. The corresponding numerical curve for Lorentzian noise is also included. It can be observed that a deviation exists but the differences are relatively small for this value of correlation time

($\varepsilon = 10^{-1}$). A two-dimensional case perpendicular to the magnetic field is shown in Figure 2. The average and variance of the radial position and velocity are estimated for coloured noise. A well-known result is observed, namely that asymptotically the variance of the position grows linearly with time. The oscillatory behaviour is due to the influence of the magnetic field.

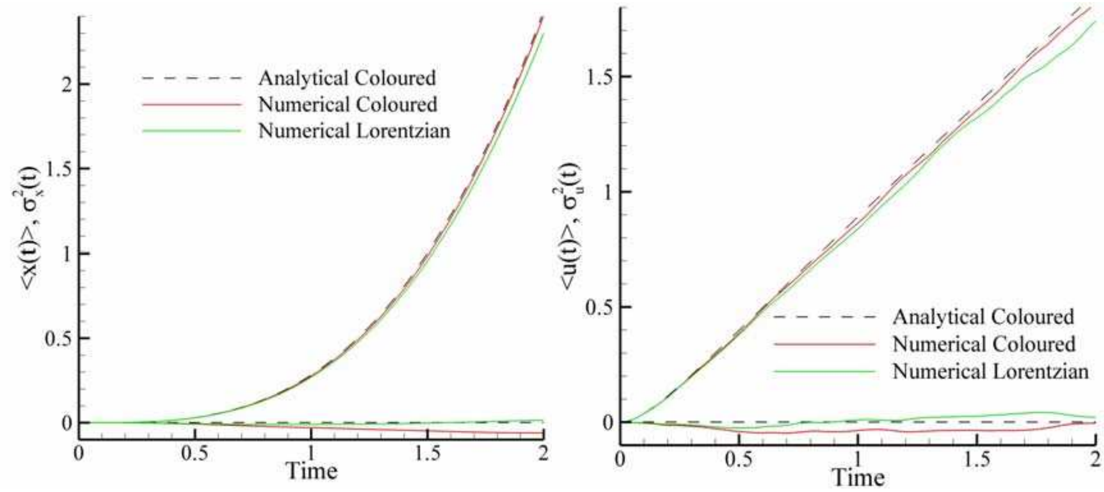

Figure 1: Mean and variance of position (left) and velocity (right) for the 1D motion along the magnetic field for coloured and Lorenzian noises with $c^2 = 1$, $\varepsilon = 10^{-1}$, $\gamma = 10^{-2}$.

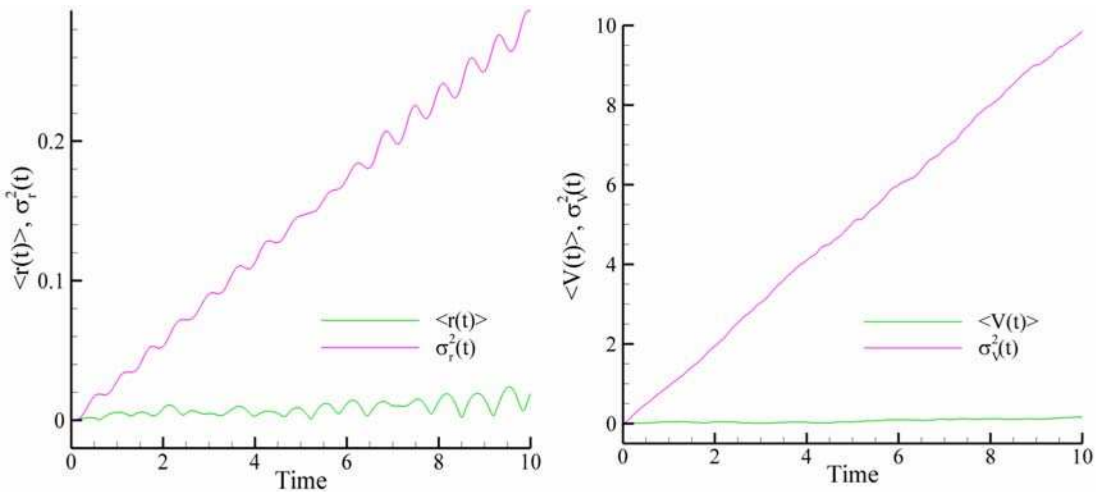

Figure 2: Mean and variance of position (left) and velocity (right) for the 2D motion transversal to the magnetic field for coloured noise with $c^2 = 1$, $\varepsilon = 10^{-1}$, $\gamma = 10^{-2}$, $\Omega = 10$.

The parametric study has shown that a variation in c does not change the results qualitatively. On the contrary, the dependence on γ is more apparent, especially in one dimension, since it may alter the curvature of the standard deviation of position significantly. The magnitude of the velocity is increased when the friction is decreased and larger times are required to obtain the long-term behaviour. The value of the correlation time ε did not affect the results considerably for the range we examined ($\varepsilon < 10^{-1}$).

Spectral simulation methods are suited to study the random motion due to stochastic electric and/or magnetic fields. Preliminary results are presented for the one-dimensional motion subject to a stochastic electric field with Gaussian correlation

$$\langle E(x)\rangle = 0 \quad \text{and} \quad \langle E(x')E(x'+x)\rangle = (\lambda\sqrt{\pi})^{-1}\exp[-x^2/\lambda^2].$$

Here, the equations of motion are

$$\frac{dx(t)}{dt} = u(t) \quad \text{and} \quad \frac{du(t)}{dt} = p^2 E(x),$$

with $u(0)=u_0$. Now, the initial velocity is important because the problem is nonlinear. When the particle is initially at rest the motion is bounded and oscillatory, while unbounded motion may occur for non-vanishing initial velocity above some threshold. The mean and variance of position and velocity, shown in Figure 3, are in agreement with this conclusion. Moreover, they will be compared with results obtained in studies using analytical approximations.

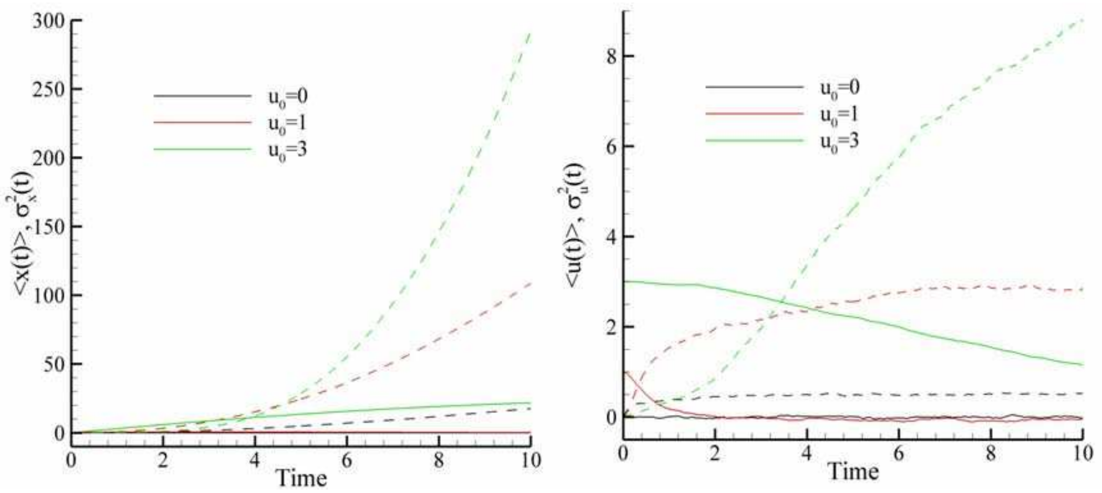

Figure 3: Mean (solid lines) and variance (dashed lines) of position (left) and velocity (right) of a charged particle moving inside a stochastic electric field for various initial velocities with $p^2 = 1$, $\lambda = 10^{-2}$.

Concluding, it should be noted that the Fourier expansion method has been successfully applied for this type of stochastic motion. The numerical codes that have been developed will also be used for the simulation of more complex problems, such as random motion in inhomogeneous external magnetic fields, including one with helicoidal field lines.

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