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1D transport equation for toroidal momentum in a tokamak

V.A. Rozhansky[1], I.Yu. Senichenkov[1], G.V. Pereverzev[2]

[1]*St. Petersburg State Polytechnical University, 195251, St. Petersburg, Russia*

[2] *Max-Planck Institut für Plasmaphysik, Euratom Association, D-85748 Garching, Germany*

1. Introduction

At present the 1D equations for particle and energy transport across the magnetic flux surfaces in a tokamak are widely used in several numerical codes such as ASTRA, TRANSP, JETTO [1-3] and others. In contrast, the transport of toroidal momentum has not been studied so intensively. In the recent years several mechanisms of the toroidal momentum transport has been put forward (see [4] and references therein). It is shown that both diffusion and convection determine the radial profile of the toroidal rotation in a tokamak.

In the present paper we suggest the 1D equation for the momentum transport with the given turbulent transport coefficients. The averaged 1D equation for toroidal rotation is derived with account of the toroidal Pfirsch-Schlüter fluxes. The averaging is based on the ambipolar condition. It is demonstrated that the account of the Pfirsch-Schlüter fluxes corresponds to the torque in the toroidal direction which is proportional to the gradient of the ion temperature. This effect is novel and has never been discussed before. The boundary condition at the separatrix and its effect on the toroidal rotation are briefly discussed.

2. Equation for the toroidal rotation

The orthogonal coordinate system (a,ϑ,ζ), Fig. 1, is used where a is a flux surface label, ϑ is the poloidal coordinate, and ζ is the toroidal angle.

The quantities h_a, h_ϑ, h_ζ are the Lame coefficients and $h_\zeta = R$, R is the major radius. The poloidal flux ψ is a flux through the shaded area (Fig 1) $\psi = 2\pi\int_{R=0}^{R_{\min}} RB_p \mathrm{d}R$.

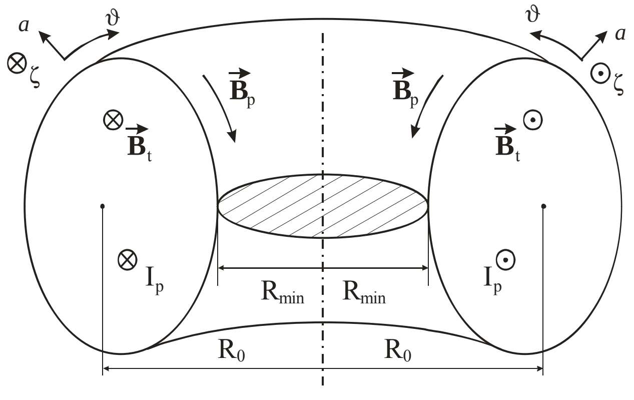

Figure 1. Flux surface in a tokamak with a coordinate system (a,ϑ,ζ).

Introducing $\psi_0 = 2\pi\int_{R=0}^{R_0} RB_p \mathrm{d}R$ – the flux at the magnetic axis – we define a new variable $\Psi = \psi_0 - \psi$. The magnetic field and its components are

$$\vec{B} = I\nabla\zeta + \frac{1}{2\pi}[\nabla\Psi\times\nabla\zeta]; \quad B_p = \frac{1}{2\pi h_a R}\frac{\partial\Psi}{\partial a}; \quad B_t = \frac{I}{R}.$$

The toroidal component of the momentum balance equation is

$$\frac{\partial}{\partial t}\left(m_i n u_\zeta\right)+\left[\nabla\cdot m_i n\vec{u}\vec{u}\right]_\zeta=\left[\vec{j}\times\vec{B}\right]_\zeta-\nabla\cdot\pi_i-\nabla\cdot\pi_i^{NEO}+F_\zeta\,, \tag{1}$$

where $\vec{F}$ is an external force, e.g. due to NBI. The net radial current I_a through the flux surface should be zero, so the ambipolarity condition reads (S is a flux surface area)

$$\left\langle\left\langle\left[\vec{j}\times\vec{B}\right]_\zeta / B_p\right\rangle\right\rangle=\left\langle\left\langle j_a\right\rangle\right\rangle=\frac{I_a}{S}=0 \tag{2}$$

where $\langle\langle\ \rangle\rangle$ denotes the surface averaging $<< f >>=\dfrac{\int f(a,\vartheta,\zeta)h_\vartheta R d\vartheta}{\int h_\vartheta R d\vartheta}$

In contrast the volume averaging is defined as

$$< f >=\frac{\int f(a,\vartheta,\zeta)\sqrt{g}\,d\vartheta}{\int\sqrt{g}\,d\vartheta}=\frac{\partial}{\partial V}\int f(a,\vartheta,\zeta)d^3\vec{r} \tag{3}$$

where $\sqrt{g}=h_a h_\vartheta h_\zeta$ is a metric tensor determinant and $V=\int_0^a\sqrt{g}\,da\,d\vartheta\,d\zeta$ is a volume inside the flux surface $a=const$.

The inertial term will be considered in the simplified form retaining only radial transport of the toroidal momentum:

$$\left\langle\left\langle\frac{1}{B_p}\left[\nabla\cdot m_i n\vec{u}\vec{u}\right]_\zeta\right\rangle\right\rangle=\left\langle\left\langle\frac{1}{B_p R\sqrt{g}}\left[\frac{\partial}{\partial a}\left(\frac{R\sqrt{g}}{h_a}\left(m_i n u_a u_\zeta\right)\right)\right]\right\rangle\right\rangle. \tag{4}$$

The viscosity on the r.h.s. of Eq. (1) is separated into turbulent viscosity $\nabla\cdot\vec{\pi}_i$ and neoclassical viscosity $\nabla\cdot\vec{\pi}_i^{NEO}$. It is known [5] that to the first order $\left\langle\left\langle\left[\nabla\cdot\pi_i^{NEO}\right]_\zeta / B_p\right\rangle\right\rangle\approx 0$.

The turbulent viscosity term is assumed to have the form

$$-\left\langle\left\langle\frac{\left[\nabla\cdot\pi_i\right]_\zeta}{B_p}\right\rangle\right\rangle=-\left\langle\left\langle\frac{1}{B_p R\sqrt{g}}\left[\frac{\partial}{\partial a}\left(\frac{R\sqrt{g}}{h_a}\left(m_i n\tilde{u}_a u_\zeta-\eta\frac{\partial u_\zeta}{h_a\partial a}\right)\right)\right]\right\rangle\right\rangle \tag{5}$$

Here η is a turbulent viscosity coefficient, and $\tilde{u}_a$ is a convective radial velocity [4] (normally directed inward), which is different from the radial particle velocity u_a. The first term could be combined with inertia term with $u_a+\tilde{u}_a=u_a^{conv}$ being net convective term.

Dividing (1) by B_p, performing surface averaging, with account of (2), we finally obtain

$$\left\langle\left\langle\frac{1}{B_p}\frac{\partial}{\partial t}\left(m_i n_i u_\zeta\right)\right\rangle\right\rangle=-\left\langle\left\langle\frac{1}{B_p R\sqrt{g}}\left[\frac{\partial}{\partial a}\left(\frac{R\sqrt{g}}{h_a}\left(m_i n u_a^{conv}u_\zeta-\eta\frac{\partial u_\zeta}{h_a\partial a}\right)\right)\right]\right\rangle\right\rangle+\left\langle\left\langle\frac{F_\zeta}{B_p}\right\rangle\right\rangle. \tag{6}$$

3. Poloidal dependence of the toroidal velocity

According to the radial momentum balance (neglecting inertia and viscosity terms)

$$u_\zeta = \frac{1}{B_p}\left(u_\vartheta B_t + E_a - \frac{1}{enh_a}\frac{\partial p_i}{\partial a}\right). \qquad (7)$$

Combining this expression with particle continuity equation $\frac{1}{\sqrt{g}}\frac{\partial}{\partial\vartheta}\left(h_a Rnu_\vartheta\right) = 0$, and assuming $n = const(\vartheta)$, one obtains (for large aspect ratio with $\left|B_p\right| << \left|B_t\right|$, see also [5])

$$u_\zeta = \frac{\langle u_\zeta B_t\rangle}{\langle B_t^2\rangle}B_t + \frac{1}{B_p}\left(\frac{1}{enh_a}\frac{\partial p_i}{\partial a} - E_a\right)\cdot\left[\frac{B_t^2}{\langle B_t^2\rangle} - 1\right] = R\langle\omega\rangle + R\omega_{\nabla T_i}\cdot\left[\frac{B_t^2}{\langle B_t^2\rangle} - 1\right], \qquad (8)$$

where $\omega = \left(\vec{u}\cdot\nabla\zeta\right) = u_\zeta / R$ is a local angular velocity, $\langle\omega\rangle = \langle u_\zeta B_t\rangle / I$, and in the last transformation for the radial electric field substituted its neoclassical expression [5]

$$E_a^{(NEO)} = \frac{\langle u_\zeta B_t\rangle B_p}{B_t} + \frac{1}{en_i h_a}\left(T_i\frac{\partial n}{\partial a} + k_T n\frac{\partial T_i}{\partial a}\right); \qquad (9)$$

with coefficient $k_T = -0.17$, 1.5 and 2.69 in the banana, plateau and Pfirsch-Schlüter regimes correspondingly, so that $\omega_{\nabla T_i}$ is a poloidally independent angular velocity proportional to ∇T_i:

$$\omega_{\nabla T_i} = 2\pi\frac{(1-k_T)}{e}\frac{\partial T_i}{\partial\Psi} = \frac{(1-k_T)}{eB_p R}\cdot\frac{1}{h_a}\frac{\partial T_i}{\partial a}. \qquad (10)$$

The first term on the r.h.s. of Eq. (8) is proportional to the average toroidal velocity and is a combination of a poloidally dependent part of the toroidal rotation and a contribution from the part of Pfirsch-Schlüter flux that closes the $\vec{E}\times\vec{B}$ poloidal drift. The second term in the r.h.s. of Eq. (8) is the part of Pfirsch-Schlüter flux which is proportional to ∇T_i.

Note that the poloidally dependent toroidal rotation is transported radially as the average toroidal rotation, and its contribution should be taken into account while performing averaging in Eq. (6).

4. Averaged equation for the toroidal rotation

It is convenient to choose $\langle\omega\rangle$ as independent variable. After averaging over the flux surface with account of Eq. (8), one obtains

$$\frac{\partial}{\partial t}\left[n\frac{\partial V}{\partial a}\langle R^2\rangle\langle\omega\rangle\right] + \frac{\partial}{\partial a}\left[R_*^2\frac{\partial V}{\partial a}\langle\omega\rangle\Gamma^{conv}\right] - \frac{\partial}{\partial a}\left[\frac{\partial V}{\partial a}\left\langle\frac{\eta R^2(\nabla a)^2}{m_i}\right\rangle\frac{\partial\langle\omega\rangle}{\partial a}\right] = S_u(a) \qquad (11)$$

The effective flux density is defined as $\Gamma^{conv} = \frac{1}{S}\int nu_a^{conv}dS = \frac{1}{S}\frac{\partial V}{\partial a}\left\langle n\left(\vec{u}^{conv}\cdot\nabla a\right)\right\rangle$, and the R_*^2 is $R_*^2 = \frac{\left\langle R^2 n\left(\vec{u}^{conv}\cdot\nabla a\right)\right\rangle}{\Gamma^{conv}}\cdot\frac{\partial V}{\partial a}\cdot\frac{1}{S}$. The Γ^{conv} is the sum of the averaged particle flux density and the averaged additional momentum flux density. The sources are

$$S_u(a) = \frac{\partial}{\partial a}\left[\left(R_*^2 - \frac{I^2}{\langle B^2\rangle}\right)\omega_{\nabla T_i}\Gamma^{conv}\right] + \frac{\partial}{\partial t}\left(n\left(\langle R^2\rangle - \frac{I^2}{\langle B^2\rangle}\right)\frac{\partial V}{\partial a}\omega_{\nabla T_i}\right)$$
$$-\frac{\partial}{\partial a}\left[\frac{\partial V}{\partial a}\left[\left\langle\frac{\eta R^2(\nabla a)^2}{m_i}\right\rangle\frac{\partial \omega_{\nabla T_i}}{\partial a} - \left\langle\frac{\eta(\nabla a)^2}{m_i}\right\rangle\frac{\partial}{\partial a}\left(\frac{I^2\omega_{\nabla T_i}}{\langle B^2\rangle}\right)\right]\right] + \frac{2\pi S}{m_i}\frac{\partial \Psi}{\partial a}\left\langle\left\langle\frac{F_z}{B_p}\right\rangle\right\rangle. \quad (12)$$

The source on the r.h.s. of the equation for the toroidal rotation besides the usual terms representing the external forces contains terms which depend on the ∇T_i. The corresponding source arises from radial transport of the part of Pfirsch-Schlüter fluxes which are proportional to the ∇T_i. In the stationary case the average toroidal rotation should cancel this torque and as a result the unbalanced toroidal rotation should be established (in the absence of other mechanisms). The sign of the effect depends on the collisionality, e.g. in the banana regime the resulting average toroidal rotation should be counter-current directed.

5. Boundary conditions

The toroidal rotation near the separatrix is defined mainly by Pfirsch-Schlüter fluxes in the SOL, which are transported inside the separatrix by the turbulent perpendicular viscosity [6]. Hence the boundary condition is defined by the SOL toroidal velocity u_ζ^{SOL} (which is co-current, and has following parametric dependence $u_\zeta^{SOL} \sim T_{sep}/B_p$ [6]) at the low field side separatrix at the equatorial midplane:

$$\langle\omega\rangle_{sep} = \left(-\omega_{\nabla T_i}\cdot\left(B_t^2/\langle B_t^2\rangle - 1\right) + u^{SOL}/R\right)\Big|_{sep}. \quad (13)$$

One may expect that if the sources can be neglected (spontaneous rotation), the value $\langle\omega\rangle_{sep}$ is transported to the core, and should have the same parametric dependence as u_ζ^{SOL}. Such type of parametric dependence is consistent with the Rice scaling [7].

Note that the Eqs. (8), (11), (12) and (13) may be generalized for the arbitrary aspect ratio case and for flux surface shape varying in time.

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