36th EPS Conference on Plasma Phys. Sofia, June 29 - July 3, 2009 ECA Vol.**33E**, P-4.061 (2009)

Non-diffusive momentum transport in sheared rotating turbulence

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Large-scale plasma motions tend to appear as a robust feature in rotating systems, often in the form of shear flows (such as circulations on the surface of planets, differential rotation in stars and galaxies or zonal flows in tokamaks). There have been accumulating evidence that large-scale shear flows as well as rotation play a crucial role in determining turbulence properties and transport, such as energy transfer or mixing. The understanding of the physical mechanism for the generation of large-scale shear flows and the complex interaction among rotation, shear flows and turbulence thus lies at the heart of the predictive theory of turbulent transport in many systems. The main feature of this type of turbulence is the appearance of non-diffusive term in the transport of angular momentum which prevents a solid body rotation from being a solution of the Reynolds equation [1, 2]. Starting from Navier-Stokes equation, it is possible to show that these fluxes arise when there is a cause of anisotropy in the system, either due to an anisotropic background turbulence [3] or due to inhomogeneity such as an underlying stratification [4].

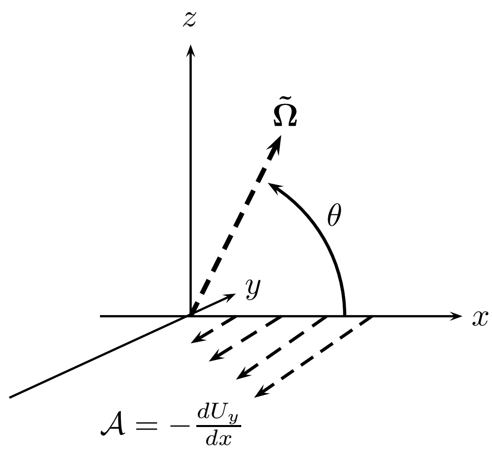

Figure 1: The configuration in our model: $\mathscr{A}$, $\tilde{\Omega}$ and θ are the shearing rate, the rotation rate and the co-latitude, respectively.

To elucidate the effect of rotation on sheared turbulence, we study single fluid MHD equations in a rotating frame (see Fig. 1) with average rotation rate $\tilde{\Omega}$ making an angle θ with a mean shear flow in the azimuthal direction: $\mathbf{U}_0 = -x\mathscr{A}\,\hat{j}$. We study the effect of this large-scale shear on the transport properties of turbulence by assuming the velocity as a sum of a radial shear (i.e. in the x-direction) and fluctuations: $\mathbf{u} = \mathbf{U}_0 + \mathbf{v} = U_0(x)\,\hat{j} + \mathbf{v} = -x\mathscr{A}\,\hat{j} + \mathbf{v}$. We resort to the quasi-linear approximation where the product of fluctuations is neglected to obtain the following equations for the evolution of the fluctuating velocity field:

$$\begin{aligned} \partial_t \mathbf{v} + \mathbf{U}_0 \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{U}_0 &= -\nabla p + \nu \nabla^2 \mathbf{v} + \mathbf{f} - \boldsymbol{\Omega} \times \mathbf{v} \,, \\ \nabla \cdot \mathbf{v} &= 0 \,, \end{aligned} \tag{1}$$

where p and $\mathbf{f}$ are respectively the small-scale components of the pressure and forcing, and $\Omega \equiv 2\tilde{\Omega}$.

As the large-scale velocity is in the y direction, we are mostly interested in the momentum transport in that direction. The equation for the (large-scale) azimuthal velocity U_0 is then given by Navier-Stokes equation with the contribution from fluctuations given by $\nabla \cdot \mathbf{R}$, where $\mathbf{R} = \langle \mathbf{v} \mathrm{v}_y \rangle$ is the Reynolds stress. One can formally Taylor expand $\mathbf{R}$ with respect to the gradient of the large-scale flow:

$$R_i = \Lambda_i U_0 - \nu_T \, \partial_x U_0 \, \delta_{i1} + \cdots = \Lambda_i U_0 + \nu_T \mathscr{A} \, \delta_{i1} + \ldots \,. \tag{2}$$

where we introduced two coefficients Λ_i and ν_T. The effect of the turbulent viscosity ν_T is simply to change the viscosity from the molecular value ν to the effective value $\nu + \nu_T$. In comparison, the first term $\Lambda_i U_0$ in Eq. (2) is proportional to the velocity rather than its gradient. This means that it does not vanish for a constant velocity field and can thus lead to the creation of gradient in the velocity field. This term is equivalent to the α-effect in dynamo theory [5], and has been known as the Λ-effect [6] or anisotropic kinetic alpha (AKA)-effect [7].

To calculate the Reynolds stress, we prescribe the forcing in Eq. (1) to be incompressible, isotropic and short correlated in time (modelled by a δ-function) with a power spectrum F. Specifically, we assume:

$$\langle \tilde{f}_i(\mathbf{k}_1, t_1) \, \tilde{f}_j(\mathbf{k}_2, t_2) \rangle = \tau_f \, (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2) \, \delta(t_1 - t_2) F(k) \, (\delta_{ij} - k_i k_j / k^2) \,. \tag{3}$$

The angular brackets stand for an average over realisations of the forcing, and τ_f is the (short) correlation time of the forcing. To obtain the Reynolds stress, we solve Eqs. (1) and use the results of these and Eq. (3) to compute the turbulent transport. Here, only the results will be provided.

We first consider the case where the direction of the rotation and the shear are orthogonal. Expanding the velocity in powers of $\bar{\Omega} = \Omega / \mathscr{A}$, we compute the turbulence amplitude and transport coefficients up to second order in $|\bar{\Omega}| \ll 1$. In the strong shear limit ($\xi \ll 1$), the transport of angular momentum can be found as a sum of two terms $\langle \mathrm{v}_x \mathrm{v}_y \rangle = \nu_T \mathscr{A} + \Lambda_x \boldsymbol{\Omega}$, the first being even and the second odd with the rotation rate. The first term, the turbulent viscosity, takes the following form:

$$\nu_T \sim \frac{\tau_f}{\mathscr{A}^2} \int \frac{d^3k}{(2\pi)^3} F(k) \left[-\frac{1}{2} + L_3(\mathbf{k}) \right] , \tag{4}$$

where L_3 is a positive definite function of $\mathbf{k}$. For $\bar{\Omega} = 0$ we recover the result of [8] showing that the turbulent viscosity is reduced proportionally to $\mathscr{A}^{-2}$ for strong shear. The turbulent viscosity can be either positive or negative depending on the relative magnitude of the two terms inside the integral. In the 2D limit (where $L_3 = 0$), we can easily see that the turbulent viscosity is negative. On the contrary, in the case of an isotropic forcing in 3D, the turbulent viscosity is positive. The correction due to the rotation is proportional to Ω and is thus odd in the rotation. This is the so-called Λ-effect, a non-diffusive contribution to Reynolds stress, which can be shown to be:

$$\Lambda_x \sim \frac{\tau_f}{\mathscr{A}^2} \int \frac{d^3k}{(2\pi)^3} F(k) L_4(\mathbf{k}) \left(-\ln \xi \right) . \tag{5}$$

It is important to emphasise that this non trivial Λ-effect results from an anisotropy induced by shear flow on the turbulence even when the driving force is isotropic. This should be contrasted to the case without shear flow where non-diffusive fluxes emerge only for anisotropic forcing [3]. In our case, the anisotropy in the velocity field is not artificially introduced in the system but is created by the shear and calculated self-consistently.

Secondly, we consider the case where the directions of rotation and shear are parallel. In that case, the turbulent viscosity is the same as previously whereas the Λ effect takes the following form:

$$\Lambda_z \sim -\frac{\tau_f}{\mathscr{A}^2} \int \frac{d^3k}{(2\pi)^3} F(k) \left(\frac{3}{2\xi} \right)^{2/3} \left[L_7(\mathbf{k}) - L_8(\mathbf{k}) \right] . \tag{6}$$

Eq. (6) shows that Λ_z is of indefinite sign, as L_7 and L_8 are two positive definite functions but appear with different signs. However, as for Eq. (5), in the case of an isotropic forcing, the function L_7 dominates over L_8, leading to a negative Λ_z. This Λ-effect arises even in the case of an isotropic forcing as the shear favours fluctuations in the y-direction compared to that in the z-direction (see [8]), leading to anisotropic turbulence in the plane perpendicular to the rotation. Eq. (6) also shows that Λ_z is larger (scaling as $\mathscr{A}^{-4/3}$) than Λ_x in the equatorial case.

To summarise, we found that the Reynolds stress involves two contributions: the turbulent viscosity and the Λ-effect. The latter, a source of non-diffusive flux, is present even with isotropic forcing due to the shear-induced anisotropy [9]. Note that for an isotropic forcing, the turbulent viscosity is found to be positive and therefore cannot act on its own as a source of differential rotation. Differential rotation can, however, be created by the Λ-effect: since it does not depend only on the gradient of angular velocity, it does not vanish for a uniform rotation and thus prevents the uniform rotation from being a solution of the large-scale momentum equation. This has interesting implications for astrophysical systems subjected to rotation. For instance, to determine the sign of the radial differential rotation of the Sun as a result of the Λ-effect, we seek a solution of the large-scale turbulent equations by demanding the Reynolds stress to vanish [4]. At the equator (where shear and rotation are perpendicular), we then obtain the following equation for the shear:

$$\nu_T(\mathscr{A})\mathscr{A} + \Lambda(\mathscr{A})\Omega = 0 \, . \tag{7}$$

As both ν_T and Λ are positive, $\mathscr{A}$ must be negative. Interestingly, this is exactly what the observation indicates (a rotation rate decreasing towards the interior at the equator). We cannot use such a simple equation to predict the sign of the shear at the pole as the Λ-effect now appears in the $\langle \mathrm{v}_y \mathrm{v}_z \rangle$ component of the Reynolds stress and is found to scale as $\mathscr{A}^{-4/3}$, which is larger than Λ_x ($\propto \mathscr{A}^{-2}$) near the equator.

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