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Lower hybrid radiation pattern from ray tracing

R.A.Cairns[1], V. Fuchs[2]

[1]*University of St Andrews, St Andrews, UK*

[2]*Association EURATOM - IPP AVČR, v.v.i, Za Slovankou 3, 18200 Prague 8, CZ*

In lower hybrid heating and current drive the antenna launching the wave produces a range of parallel wavenumbers and rays from different parts of the spectrum may follow divergent trajectories. If most of the energy is concentrated around a single dominant wavenumber, then following a single ray may be adequate, or methods which seek to look at the spreading of the ray due to diffraction might be used [1]. However, particularly in a smaller tokamak like COMPASS which is now operational in Prague, there may be a wide spread of wavenumbers carrying a substantial part of the power. Here we develop a method to predict the way in which this power propagates by following multiple rays. The question then is how to combine these ray paths into an accurate representation of the power flow within the tokamak. We will first show through a simple example that a method of stationary phase can give a good approximation to the wave field, discuss how this might be extended to the tokamak problem and finally look at a slightly more complicated example, still amenable to exact solution, and demonstrate how our technique can be applied to it and give good results.

We begin by considering the problem of the two-dimensional wave equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\frac{\omega^2}{c^2}\phi$$

on the region $y \geq 0$ with the boundary condition $\phi(x,0) = e^{ik_0 x}$ if $-a < x < a$ and zero otherwise. The solution is readily found to be

$$\phi(x,y) = \int_{-\infty}^{\infty} \frac{\sin\left((k-k_0)a\right)}{(k-k_0)a} \exp\left(ikx + i\sqrt{1-k^2}\,y\right) dk\,,$$

lengths having been scaled to c/ω, and applying the stationary phase approximation to the integral gives

$$\left|\phi(x,y)\right| = \left| \frac{\sin\left(\dfrac{xa}{\sqrt{x^2+y^2}} - k_0\right)}{\dfrac{xa}{\sqrt{x^2+y^2}} - k_0} \sqrt{2\pi \frac{y}{\left(x^2+y^2\right)^{3/2}}} \right|$$

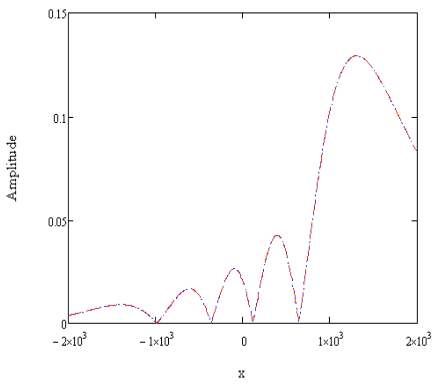

Fig1. Exact (red) and approximate (blue) amplitudes.

As can be seen from Fig.1 the approximation is excellent. It can be interpreted as being given by taking the ray path from the centre of the antenna to the point of interest, with the amplitude of the appropriate Fourier component, then multiplying it by the factor coming from the phase integral. Of course, in the case of the tokamak, we do not have an explicit solution in terms of an integral, so we must try to apply this idea without this knowledge.

We use the fact that in a tokamak the toroidal wavenumber is conserved. We will let a variable p characterise the toroidal spectrum so that $\mathbf{k}$ has only two variable components. As in the above example, the amplitude at any point is a superposition of Fourier components with all values of p, but only those close to the value on the ray from the antenna to the point contribute significantly. If p_0 is this value and $\Phi(p)$ is the phase of the spectral component p, then

$$\Phi(p) \approx \Phi(p_0) + \frac{1}{2}\Phi''(p_0)(p-p_0)^2$$

and adding contributions of different phase gives

$$\left|A(\mathbf{r})\right| \approx A_0\sqrt{\frac{2\pi}{\left|\Phi''(p_0)\right|}}$$

with A_0 the amplitude of the central Fourier component. Our problem is to find the derivative of the phase without having explicit knowledge of the latter.

The phase is given by $\Phi(p) = \int \mathbf{k}(p).d\mathbf{r}$ with the integral along the ray path from the centre of the antenna. So $\frac{d\Phi}{dp} = \int \frac{d\mathbf{k}}{dp}.d\mathbf{r} = -\int \frac{d\mathbf{k}}{dp}.\frac{\partial D}{\partial \mathbf{k}}d\tau$ with $d\tau = \frac{dt}{\frac{\partial D}{\partial \omega}}$.

Now, $\frac{d\mathbf{k}}{dp}.\frac{\partial D}{\partial \mathbf{k}} = \frac{dD}{dP} = 0$ since D is identically zero, and so $\frac{d\Phi}{dp} = 0$, confirming that the ray path is, indeed, given by the condition of stationary phase. The second derivative can be transformed, using the relation,

$$0 = \frac{d^2D}{dp^2} = \frac{d}{dp}\left(\frac{d\mathbf{k}}{dp}.\frac{\partial D}{\partial \mathbf{k}}\right) = \frac{d^2D}{dp^2}.\frac{\partial D}{\partial \mathbf{k}} + \frac{d\mathbf{k}}{dp}.\frac{\partial^2 D}{\partial \mathbf{k}\partial \mathbf{k}}.\frac{d\mathbf{k}}{dp}$$

to give

$$\frac{d^2\Phi}{dp^2} = \int \frac{d\mathbf{k}}{dp} \cdot \frac{\partial^2 D}{\partial\mathbf{k}\partial\mathbf{k}} \cdot \frac{\partial\mathbf{k}}{dp} d\tau \,. \qquad (1)$$

The derivative of D can readily be found, so we need now the derivative of **k**. We can do so as follows. From the ray tracing equations,

$$\frac{d\mathbf{k}}{d\tau} = \frac{\partial D}{\partial\mathbf{r}}$$

and since we want to follow the same spatial path, but with varying values of *p,* we express this as a spatial derivative, using

$$\frac{d\mathbf{k}}{d\tau} = \left(\frac{d\mathbf{r}}{d\tau} \cdot \frac{d}{d\mathbf{r}}\right)\mathbf{k} = \left(\frac{\partial D}{\partial\mathbf{k}} \cdot \frac{d}{d\mathbf{r}}\right)\mathbf{k} = \frac{\partial D}{\partial\mathbf{r}} \qquad (2)$$

giving an equation for the rate of change of **k** along the ray path. To find the derivative with respect to *p* we can repeat this with the same path, but an adjacent value of *p.* This will change the initial value of **k** as well as altering the integrand along the path.

To illustrate this procedure we take a problem in two dimensions which does not simply have the straight rays of our initial problem, but is analytically tractable. The problem we consider is to solve the equation $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\frac{\omega^2}{c^2}\left(1 + ay\right)\phi$ with the same boundary condition as before. If we Fourier transform in *x* with *p* the transform variable, which therefore characterises the spectrum of launched waves, then the solution is

$$\phi = \int_{-\infty}^{\infty} F(p) \left[\frac{iAi\left(a^{2/3}\left(1 + ay - p^2\right)\right) + Bi\left(a^{2/3}\left(1 + ay - p^2\right)\right)}{iAi\left(a^{2/3}\left(1 - p^2\right)\right) + Bi\left(a^{2/3}\left(1 - p^2\right)\right)}\right] e^{ipx} dp$$

with $F(p) = \frac{\sin\left(\left(p - k_0\right)a\right)}{\pi(p - k_0)}$ and *Ai* and *Bi* Airy functions. If we let *k* be the *y*-component of the wavenumber, then the local dispersion relation is $D = 1 + ay - p^2 - k^2 = 0$ and the analogue of (2) is $2k\frac{dk}{dy} = a$. Inserting *k* as a function of the position along the ray we can carry out the integration easily and just recover the result $k = \sqrt{1 + ay - p^2}$.

We can then, of course, find the derivative of k with respect to p explicitly and the integrand in (1) becomes $\frac{-2p^2}{1+ay-p^2}$. Using $\frac{dy}{d\tau} = -\frac{\partial D}{\partial k} = 2\sqrt{1+ay}$ we can make the integration variable y and evaluate the integral to give

$$\Phi'' = \frac{2}{a}\left[\left(1+ay-p^2\right)^{1/2} - \left(1-p^2\right)^{1/2}\right] + \frac{p^2}{a}\left[\frac{1}{\left(1+ay-p^2\right)^{1/2}} - \frac{1}{\left(1-p^2\right)^{1/2}}\right].$$

Note that while these results are obtained analytically, we do not use the analytic solution and we do not have an explicit phase integral to approximate. A further correction we must make is to take account of varying group velocity, by introducing a factor inversely proportional to its square root. Typical results then obtained are shown below.

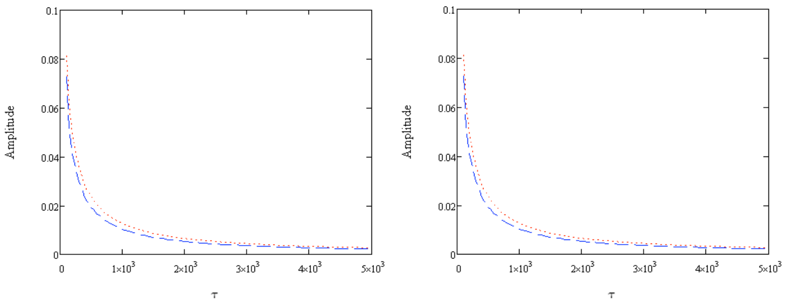

Fig 2. A comparison between the approximate and exact wave amplitudes along two ray paths with two initial values of p (0.25 on the left and 0.75).

To conclude, we have shown how a phase integral approximation gives good results for a simple wave problem then outlined how the same method might be applied to find the wave amplitude for a tokamak. A further example is developed to show how this procedure can be implemented in a problem where the rays are not simply straight lines, but which still allows comparison with an analytic solution. The important point to note is that the calculation does not require explicit knowledge of the solution and, while in this case the various integrals involved can be done analytically, they could easily be done numerically. Our next step will be to implement the procedure for a tokamak. Clearly enough rays will be needed to cover the significant parts of the energy spectrum, but the procedure should be computationally simpler than doing a full wave solution.

References

1. . A.G. Peeters, Phys. Plasmas **3,** 4386 (1996) and references therein.