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Particle orbits in an axisymmetric equilibrium plasma

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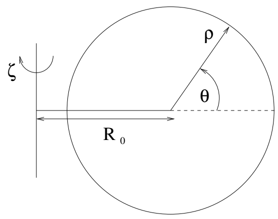
Shafranov coordinate system



The (ρ,θ,ζ) coordinate system in the Figure is usually used when working with tokamaks. The Shafranov coordinates are defined as the sinistral system (ρ,θ,ϕ), where $\phi = -\zeta$, so that the position vector for a point on a toroidal configuration is given by

$$\mathbf{r} = (R_0 + \rho\cos\theta - \triangle(\rho))\cos\phi\,\hat{\mathbf{i}} + (R_0 + \rho\cos\theta - \triangle(\rho))\sin\phi\,\hat{\mathbf{j}} + \rho\sin\theta\,\hat{\mathbf{k}}. \tag{1}$$

The $\triangle(\rho)$ represents the Shafranov shift in the toroidal plane towards the major axis. Using the notation

$$R = R_0 + \rho\cos\theta - \triangle(\rho), \tag{2}$$

one obtains from the position vector

$$\frac{\partial\mathbf{r}}{\partial\rho} = \mathbf{r}_\rho = \Big[\big(\cos\theta - \triangle'\big)\cos\phi,\, \big(\cos\theta - \triangle'\big)\sin\phi,\, \sin\theta\Big], \tag{3}$$

$$\frac{\partial\mathbf{r}}{\partial\theta} = \mathbf{r}_\theta = \rho\,[-\sin\theta\cos\phi,\, -\sin\theta\sin\phi,\, \cos\theta\,], \tag{4}$$

$$\frac{\partial\mathbf{r}}{\partial\phi} = \mathbf{r}_\phi = R\,[-\sin\phi,\, \cos\phi,\, 0\,], \tag{5}$$

so that the unit vectors are given by

$$\mathbf{e}_\rho = \frac{\mathbf{r}_\rho}{|\mathbf{r}_\rho|} = \frac{1}{1-\triangle'\cos\theta}\Big[\big(\cos\theta - \triangle'\big)\cos\phi,\, \big(\cos\theta - \triangle'\big)\sin\phi,\, \sin\theta\Big], \tag{6}$$

$$\mathbf{e}_\theta = \frac{\mathbf{r}_\theta}{|\mathbf{r}_\theta|} = [-\sin\theta\cos\phi,\, -\sin\theta\sin\phi,\, \cos\theta\,], \tag{7}$$

$$\mathbf{e}_\phi = \frac{\mathbf{r}_\phi}{|\mathbf{r}_\phi|} = [-\sin\phi,\, \cos\phi,\, 0\,]. \tag{8}$$

The Jacobian is defined to be

$$J = \mathbf{r}_\rho\cdot(\mathbf{r}_\theta\wedge\mathbf{r}_\phi) = -\rho R(1-\triangle'\cos\theta). \tag{9}$$

The volume element for this system is given by

$$dV = \mathbf{r}_\rho\cdot(\mathbf{r}_\theta\wedge\mathbf{r}_\phi) = J, \tag{10}$$

with the understanding that integration takes place in ρ from 0 to the radius of the toroidal surface, $\theta \in [0, 2\pi]$, and ϕ from 2π to 0. The following (non-orthogonal) properties are noted:

$$\mathbf{r}_\rho \cdot \mathbf{r}_\theta = \rho \triangle' \sin\theta\,; \qquad \mathbf{r}_\rho \cdot \mathbf{r}_\phi = 0\,; \qquad \mathbf{r}_\theta \cdot \mathbf{r}_\phi = 0\,. \tag{11}$$

The line element can be written as

$$ds^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j = (1 - 2\triangle' \cos\theta)\,\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\theta^2 + R^2\,\mathrm{d}\phi^2 + 2\rho\triangle' \sin\theta\,\mathrm{d}\rho\mathrm{d}\theta, \tag{12}$$

where the coordinates (x^1, x^2, x^3) correspond to (ρ, θ, ϕ).

Equations of motion

The velocity is written as

$$u^2 = \left(\frac{ds}{dt}\right)^2 = g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = (1 - 2\triangle' \cos\theta)\,\dot\rho^2 + 2\rho\triangle' \sin\theta\,\dot\rho\dot\theta + \rho^2\,\dot\theta^2 + R^2\,\dot\phi^2, \tag{13}$$

so that the Lagrangian becomes

$$\begin{aligned} L &= \frac{m}{2}\left(\frac{ds}{dt}\right)^2 - U \\ &= \frac{m}{2}\left[(1 - 2\triangle' \cos\theta)\,\dot\rho^2 + 2\rho\triangle' \sin\theta\,\dot\rho\dot\theta + \rho^2\,\dot\theta^2 + R^2\,\dot\phi^2\right] - U(\rho,\theta,\phi) \end{aligned} \tag{14}$$

with

$$R^2 = R_0^2 + \rho^2\cos^2\theta + 2R_0\rho\cos\theta - 2R_0\triangle - 2\triangle\rho\cos\theta \tag{15}$$

and where we have used the approximation

$$\triangle^2 \ll 1. \tag{16}$$

We proceed by writing the equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot x^i}\right) - \frac{\partial L}{\partial x} = 0 \tag{17}$$

for each coordinate in the system. For the ρ coordinate

$$\begin{aligned} &(1 - 2\triangle' \cos\theta)\ddot\rho - \triangle''\dot\rho^2\cos\theta + 2\triangle'\,\dot\rho\dot\theta\sin\theta \\ &+ \triangle'\rho\ddot\theta\sin\theta - \rho\dot\theta(1 - \triangle'\cos\theta) - R\dot\phi(\cos\theta - \triangle') \;=\; \frac{-1}{m}\frac{\partial U}{\partial \rho}\,. \end{aligned} \tag{18}$$

Approximation (16) allows us to write $(1 - 2\triangle'\cos\theta) = 1 - 2\triangle'\cos\theta + (\triangle'\cos\theta)^2$ $= (1 - \triangle'\cos\theta)^2$ as well as to multiply selectively with the factor $[1 - (\triangle'\cos\theta)^2]$, so that (18) becomes

$$\begin{aligned} &(1 - \triangle'\cos\theta)\Big[(1 - \triangle'\cos\theta)\ddot\rho + (1 + \triangle'\cos\theta)(2\triangle'\,\dot\rho\dot\theta\sin\theta + \triangle'\rho\ddot\theta\sin\theta - \triangle''\dot\rho^2\cos\theta) \\ &\qquad - \rho\dot\theta^2 - R\dot\phi(\cos\theta - \triangle')(1 + \triangle'\cos\theta)\Big] = \frac{-1}{m}\frac{\partial U}{\partial \rho}\,. \end{aligned} \tag{19}$$

Again, (16) allows us to write $(\cos\theta - \triangle')(1+\triangle'\cos\theta) = \cos\theta - \triangle'\sin^2\theta$, so that (19) becomes

$$(1-\triangle'\cos\theta)\ddot{\rho} + 2\triangle'\dot{\rho}\dot{\theta}\sin\theta + \triangle'\rho\ddot{\theta}\sin\theta - \triangle''\dot{\rho}^2\cos\theta - \rho\dot{\theta}^2 - R\dot{\phi}(\cos\theta - \triangle'\sin^2\theta) = \frac{-1}{m}\frac{1}{1-\triangle'\cos\theta}\frac{\partial U}{\partial \rho}, \quad (20)$$

using the approximations (16) and $\triangle'\triangle'' \ll 1$. The θ component of the equation of motion (17) is written as

$$\triangle''\dot{\rho}^2\sin\theta + \triangle'\ddot{\rho}\sin\theta + 2\dot{\rho}\dot{\theta} + \rho\ddot{\theta} + R\dot{\phi}\sin\theta = \frac{-1}{m\rho}\frac{\partial U}{\partial \theta}, \quad (21)$$

while

$$\begin{aligned} \frac{d}{dt}\left[R^2\dot{\phi}\right] &= \ddot{\phi}R^2 + \dot{\phi}\left[2R\dot{\rho}\cos\theta - 2R\rho\dot{\theta}\sin\theta - 2\triangle'\dot{\rho}(R_0+\rho\cos\theta)\right] \\ &= \ddot{\phi}R^2 + 2R\dot{\phi}(\dot{\rho}\cos\theta - \rho\dot{\theta}\sin\theta - \triangle'\dot{\rho}) \end{aligned} \quad (22)$$

using notation (15) as well as the approximation $\triangle\triangle' \ll 1$, so that the ϕ component of (17) becomes

$$\ddot{\phi}R^2 + 2R\dot{\phi}(\dot{\rho}\cos\theta - \rho\dot{\theta}\sin\theta - \triangle'\dot{\rho}) = \frac{-1}{m}\frac{\partial U}{\partial \phi}. \quad (23)$$

The set of equations to be solved is given by (20), (21) and (23). The integral of energy

$$E = \dot{x}^i\frac{\partial L}{\partial \dot{x}^i} - L \quad (24)$$

is given by

$$E = \frac{m}{2}\left[(1-2\triangle'\cos\theta)\dot{\rho}^2 + 2\triangle'\rho\dot{\rho}\dot{\theta}\sin\theta + \rho^2\dot{\theta}^2 + R^2\dot{\phi}^2\right] + U(\rho,\theta,\phi). \quad (25)$$

Under axisymmetry in the toroidal direction the ϕ coordinate becomes ignorable, so that the ϕ component of the equation of motion (17) becomes

$$\frac{d}{dt}\left[R^2\dot{\phi}\right] = 0, \qquad R^2\dot{\phi} = P_\phi, \quad (26)$$

where P_ϕ is a constant. This expresses the conservation of angular momentum in the toroidal direction. Also, under toroidal axisymmetry, no ϕ dependence means that equation (23) falls away, so that the set of equations of motion reduces to only the ρ and θ components, given by the simplified (20) and (21):

$$(1-\triangle'\cos\theta)\ddot{\rho} + 2\triangle'\dot{\rho}\dot{\theta}\sin\theta + \triangle'\rho\ddot{\theta}\sin\theta - \triangle''\dot{\rho}^2\cos\theta - \rho\dot{\theta}^2 = \frac{-1}{m}\frac{1}{1-\triangle'\cos\theta}\frac{\partial U}{\partial \rho}, \quad (27)$$

$$\triangle''\dot{\rho}^2\sin\theta + \triangle'\ddot{\rho}\sin\theta + 2\dot{\rho}\dot{\theta} + \rho\ddot{\theta} = \frac{-1}{m\rho}\frac{\partial U}{\partial \theta}. \quad (28)$$

Numerical implementation

By introducing the two variables

$$r = \dot{\rho}, \qquad\qquad \tau = \dot{\theta}, \tag{29}$$

the toroidal axisymmetric set of equations (27) and (28) reduce to

$$\begin{pmatrix} 1-\triangle'\cos\theta & \rho\,\triangle'\sin\theta \\ \triangle'\sin\theta & \rho \end{pmatrix}\begin{pmatrix} \dot{r} \\ \dot{\tau} \end{pmatrix} = \begin{pmatrix} \rho\tau^2 - 2\,\triangle' r\tau\sin\theta + \triangle'' r^2\cos\theta - \dfrac{1}{m(1-\triangle'\cos\theta)}\dfrac{\partial U}{\partial\rho} \\ -\triangle'' r^2\sin\theta - 2r\tau - \dfrac{1}{m\rho}\dfrac{\partial U}{\partial\theta} \end{pmatrix}, \tag{30}$$

together with

$$\frac{d}{dt}\begin{pmatrix} \rho \\ \theta \end{pmatrix} = \begin{pmatrix} r \\ \tau \end{pmatrix}. \tag{31}$$

The determinant of the matrix of the left of (30) is

$$\begin{vmatrix} 1-\triangle'\cos\theta & \rho\,\triangle'\sin\theta \\ \triangle'\sin\theta & \rho \end{vmatrix} = \rho(1-\triangle'\cos\theta), \tag{32}$$

using the fact that $(\triangle')^2 \ll 1$, which allows us to calculate the inverse of the matrix on the left of (30). Matrix system (30) can now be written as

$$\frac{d}{dt}\begin{pmatrix} r \\ \tau \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}, \tag{33}$$

where

$$\begin{aligned} A &= \frac{1}{1-\triangle'\cos\theta}\Big(\rho\tau^2 - 2\,\triangle' r\tau\sin\theta + \triangle'' r^2\cos\theta\Big) - \\ &\qquad \frac{1}{m(1-2\,\triangle'\cos\theta)}\frac{\partial U}{\partial\rho} + \frac{\triangle'\sin\theta}{1-\triangle'\cos\theta}\Big(2r\tau + \frac{1}{m\rho}\frac{\partial U}{\partial\theta}\Big), &(34)\\ B &= \frac{-\triangle'\tau^2\sin\theta}{1-\triangle'\cos\theta} + \frac{\triangle'\sin\theta}{1-2\,\triangle'\cos\theta}\frac{1}{m\rho}\frac{\partial U}{\partial\rho} - \frac{1}{\rho}\Big(\triangle'' r^2\sin\theta + 2r\tau + \frac{1}{m\rho}\frac{\partial U}{\partial\theta}\Big), &(35)\end{aligned}$$

which is solved together with the system (31). To obtain the above result we have used the approximations $(\triangle')^2 \ll 1$ and $\triangle'' \ll 1$

Future work

By working in the non-orthogonal Shafranov coordinate system, the flux surfaces inside the toroidal chamber are approximated by circles shifted in the radial direction towards the major axis. We have written the components of the equation of motion and the intention is to integrate them numerically to obtain the particle orbits inside the chamber.