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Numerical solution of the linear dispersion relation in a relativistic pair plasma

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Introduction

Relativistic plasmas, in which the streaming and/or thermal speeds of the constituent particles are comparable to c play an important role in astrophysical objects such as gamma-ray bursts and pulsar winds. In order to understand the nature of these objects it is, therefore, necessary to study the various instabilities which are present in, for example, current sheets and shock fronts in such plasmas [1, 2]. Because relatively few analytic results are available, we have designed an algorithm that computes the linear dispersion relation of waves and instabilities in relativistic plasmas within a Vlasov-Maxwell description.

The method used is based on that presented by Daughton [3], and involves explicit integration of particle orbits along the unperturbed trajectories. We modify and extend this method by changing the manner in which the roots of the dispersion relation are located and adopting a fully relativistic approach. We first check our results against the dispersion curves for a single component magnetised plasma and for an unmagnetised plasma with cool counter-streaming components in the non-relativistic case. We then present results for the growth rate of the Weibel or two-stream instability in a hot unmagnetised pair plasma consisting of counter-streaming relativistic Maxwellians.

Derivation of the eigenvalue system

We start from the well-known relativistic Vlasov-Maxwell equations for each species "s" :

$$\frac{\partial f_s}{\partial t} + \vec{v} \cdot \frac{\partial f_s}{\partial \vec{r}} + q_s\,(\vec{E} + \vec{v} \wedge \vec{B}) \cdot \frac{\partial f_s}{\partial \vec{p}} = 0 \tag{1}$$

with the distribution function denoted by $f_s(\vec{r},\vec{p},t)$ and adopting the usual notations, namely $t,\vec{r},\vec{v},\vec{p},m_s,q_s,\vec{E}$ and $\vec{B}$ for respectively the time, position, 3-velocity, 3-momentum, mass and charge of a particle and the electromagnetic field. We linearise (1) for each species about an equilibrium state f_{0s} consisting of a static magnetic field aligned with the z-axis and depending only on x such that $\vec{B} = B_0(x)\,\vec{e}_z$ (we use Cartesian coordinates (x,y,z)). Introducing the electromagnetic potentials $(\phi,\vec{A})$, such that $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla} \wedge \vec{A}$ and expanding all quantities ψ in terms of plane waves with real wavenumber vector, $\vec{k} = k_y\,\vec{e}_y + k_z\,\vec{e}_z$, and complex frequency ω as $\psi(\vec{r},t) = \tilde{\psi}(x)\,e^{i\,(k_y y + k_z z - \omega t)}$ we arrive by the usual method at expressions for the charge and current densities:

$$\begin{aligned} \rho(\vec{r},t) &= \sum_s \frac{\Gamma_s\,\omega_{ps}^2\,\varepsilon_0}{\Theta_s\,c^2}\,\Big[U_{\rm s}A_{\rm y}(\vec{r},t)-\phi(\vec{r},t) \\ &+ i\,(\omega-k_{\rm y}\,U_{\rm s})\iiint \hat{f}_{0s}\int_{-\infty}^{\tau(t)}\left(\frac{\vec{p}\,'}{m_s}\cdot\vec{A}(\vec{r}\,',\tau')-\gamma'\,\phi(\vec{r}\,',\tau')\right)d\tau'\,d^3\vec{p}\Big] \end{aligned} \tag{2}$$

$$\begin{aligned} \vec{j}(\vec{r},t) &= \sum_s \frac{\Gamma_s\,\omega_{ps}^2\,\varepsilon_0}{\Theta_s\,c^2}\left[(U_{\rm s}A_{\rm y}(\vec{r},t)-\phi(\vec{r},t)))\iiint \frac{\vec{p}\,c^2}{E}\,\hat{f}_{0s}\,d^3\vec{p}\right. \\ &+ \left. i\,(\omega-k_{\rm y}\,U_{\rm s})\iiint \frac{\vec{p}\,c^2}{E}\,\hat{f}_{0s}\int_{-\infty}^{\tau(t)}\left(\frac{\vec{p}\,'}{m_s}\cdot\vec{A}(\vec{r}\,',\tau')-\gamma'\,\phi(\vec{r}\,',\tau')\right)d\tau'\,d^3\vec{p}\right] \end{aligned} \tag{3}$$

where $\gamma' = \sqrt{1+p'^2/m_s^2\,c^2}$ and $\Theta_s = k_B\,T_s/m_s\,c^2$. The proper time is denoted by $d\tau' = dt'/\gamma'$. The plasma frequency corresponding to species "s" is $\omega_{ps}^2 = n_{0s}\,q_s^2/m_s\,\varepsilon_0$ and the normalised distribution function is defined by $f_{0s} = n_{0s}\,\hat{f}_{0s}$. The trajectories are integrated over the unperturbed orbits, in the proper frame, following the equations of motion:

$$d\vec{r}\,'/d\tau' = \vec{p}\,'(\tau')/m_s \quad ; \quad d\vec{p}\,'/d\tau' = \vec{p}\,'(\tau')\wedge\vec{\omega}_B(\vec{r}\,') \tag{4}$$

with initial conditions $\vec{r}\,'(\tau'=\tau)=\vec{r}$ and $\vec{p}\,'(\tau'=\tau)=\vec{p}$. The cyclotron frequency is given by $\vec{\omega}_{B_s} = q_s\,\vec{B}/m_s$. According to equation (4), both energy γ' and the momentum component p_z' are conserved so that $z' = z_0 + p_z\,\tau'/m_s$. Consequently, the p_z integration can be performed analytically leading to (for brevity we give the full expressions only for the charge density)

$$\tilde{\rho}(x) = \sum_s \frac{\Gamma_s\,\omega_{ps}^2\,\varepsilon_0}{\Theta_s\,c^2}\left[-\tilde{\phi}(x)+i\,(\omega-k_{\rm y}\,U_{\rm s})\iint_{R^2}\hat{F}_{0s}(x,p_x,p_y)\,S_\phi(x,p_x,p_y)\,dp_x\,dp_y\right] \tag{5}$$

where we introduced the following functions :

$$\hat{F}_{0s}(x,p_x,p_y) = (4\,\pi\,m_s^3\,c^3\,\Theta_s\,K_2(1/\Theta_s))^{-1}\,e^{\Gamma_s\,U_s\,p_y/\Theta_s\,m_s\,c^2} \tag{6}$$

$$S_\phi(x,p_x,p_y) = \int_0^{-\infty}\left\{\tilde{\phi}'\,I_{\rho_\phi}-\left[(p_x'\tilde{A}_x'+p_y'\tilde{A}_y')\,\frac{I_{\rho_1}}{m_s}+\tilde{A}_z'\,I_{\rho_{A_z}}\right]\right\}e^{i\vec{k}\cdot\vec{r}\,'}\,d\tau' \tag{7}$$

$$I_{\rho_1} = m_s\,c\,\gamma_\perp\left(\sqrt{\frac{\beta}{\alpha}}+\sqrt{\frac{\alpha}{\beta}}\right)K_1(2\sqrt{\alpha\,\beta}) \tag{8}$$

$$I_{\rho_\phi} = m_s\,c\,\gamma_\perp^2\left[K_0(2\sqrt{\alpha\,\beta})+\frac{1}{2}\left(\frac{\beta}{\alpha}+\frac{\alpha}{\beta}\right)K_2(2\sqrt{\alpha\,\beta})\right] \tag{9}$$

$$I_{\rho_{A_z}} = m_s\,c^2\,\gamma_\perp^2\,\frac{1}{2}\left(\frac{\beta}{\alpha}-\frac{\alpha}{\beta}\right)K_2(2\sqrt{\alpha\,\beta}) \tag{10}$$

and the argument of the modified Bessel functions (for propagation along $\vec{e}_z$) are given by :

$$\gamma_\perp = \sqrt{1+p_\perp^2/m_s^2\,c^2} \tag{11}$$

$$\alpha = \gamma_\perp\sqrt{\Gamma_s/\Theta_s+i\,(\omega-k_z\,c)\,\tau'}/2 \tag{12}$$

$$\beta = \gamma_\perp\sqrt{\Gamma_s/\Theta_s+i\,(\omega+k_z\,c)\,\tau'}/2 \tag{13}$$

Using the Lorentz gauge, the eigenvalue system is found by inserting (2) and (3) into the wave equations

$$\frac{d^2\tilde{\phi}}{dx^2}(x) - \left(k^2 - \frac{\omega^2}{c^2}\right)\tilde{\phi}(x) + \frac{\tilde{\rho}(x)}{\varepsilon_0} = 0 \qquad (14)$$

$$\frac{d^2\tilde{\vec{A}}}{dx^2}(x) - \left(k^2 - \frac{\omega^2}{c^2}\right)\tilde{\vec{A}}(x) + \mu_0\,\tilde{\vec{j}}(x) = 0 \qquad (15)$$

The result can be written in terms of the unknown 4-dimensional vector $\vec{\Psi} = (\tilde{\phi}, \tilde{\vec{A}})$:

$$M(\omega, \vec{k}) \cdot \vec{\Psi} = 0 \qquad (16)$$

and is a non-linear eigenvalue problem for the matrix M with eigenvector $\vec{\Psi}$ and eigenvalue ω. Daughton solved this system by locating the zeroes of the determinant of M. We modify this slightly by solving simultaneously for both the eigenvalues and the eigenvectors using a Newton-Raphson algorithm which attempts to zero the ratio between the smallest and largest of the eigenvalues λ_i of the matrix equation $M \cdot \vec{u} = \lambda\,\vec{u}$. This procedure allows us to track the dispersion curves through crossing points.

Results

Fig. 1 presents comparisons of results found with a non-relativistic version of the code, showing the advantages of implementing our modified root finding algorithm. The left panel shows numerical results (dots) for non-relativistic parallel propagating oscillations of a single component plasma in a homogeneous magnetic field superimposed on the analytical expressions for the electrostatic (green), left and right-handed polarised electromagnetic modes (red and blue).

The right panel shows numerical results (red points) for a counter-streaming pair plasma with relatively small thermal spread, $v_{\rm th}/c = 10^{-3}$, compared to the drift speed $U_{\rm s} = 0.1\,c$ and superimposed on the analytic expression (blue curve) and two asymptotic expressions for zero temperature plasma (green lines).

We show this dispersion curve for the case of very small drift speeds in the left panel of Fig. 2. Here, the large k_z asymptote is absent due to thermal effects. Finally we computed the dispersion relation for a two-component fully relativistic counter-streaming plasma with high temperature, $\Theta_s = 10^2$ and different drift speeds, right panel in Figure 2. Due to the spread in momentum arising from finite temperature, the instability is suppressed for large wavenumbers k_z in both cases, classical and relativistic.

Conclusion

We have constructed an algorithm to solve the dispersion relation for non-relativistic and relativistic multi-component plasmas. The code has been validated by comparing the results

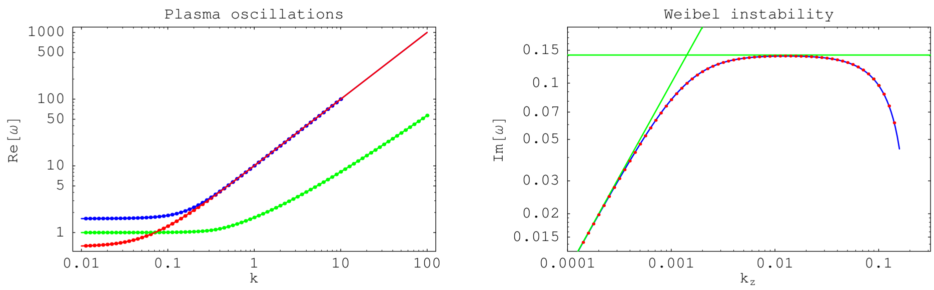

Figure 1: Dispersion relation for the transverse and longitudinal modes in a non-relativistic magnetised plasma ($\Theta_s \ll 1$), on the left, and for the Weibel instability on the right. Wavenumber is in units of $\omega_{\rm p}/v_{\rm th}$ and frequency in units of $\omega_{\rm p}$.

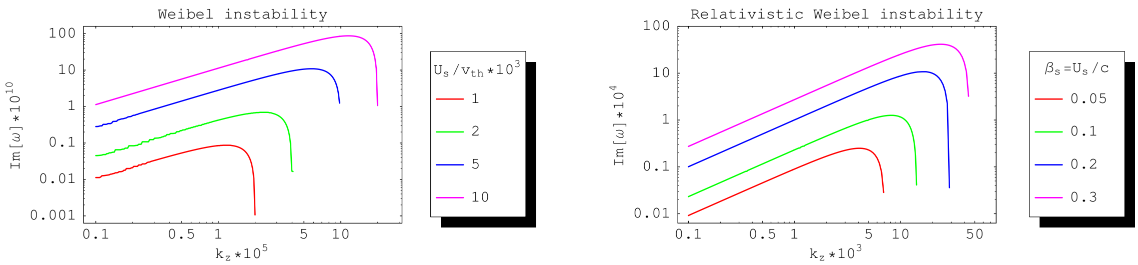

Figure 2: Growth rate for the non-relativistic ($\Theta_s \ll 1$) and relativistic ($\Theta_s \gg 1$) Weibel instability for different drift speeds $U_{\rm s}$.

with some typical configurations with homogeneous magnetic field, for which the analytical dispersion relation is known. Results have been obtained for the growth rate of the relativistic Weibel or two-stream instability which complement those found using a water-bag distribution [1, 2, 4, 5]. It is planned to use the code to study the stability properties of more general equilibrium geometries containing an inhomogeneous external background magnetic field.

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