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PHOENIX: MHD spectra of rotating laboratory and astrophysical plasmas

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Introduction

Different kinds of tokamak experiments have been done where the plasma rotates mainly in the toroidal direction. Rotation has been induced by e.g. neutral beam injection (NBI). Furthermore, in astrophysics the plasma in accretion disks also rotates mainly in the toroidal direction.

We present the newly developed spectral code PHOENIX which makes use of the Jacobi-Davidson subspace iteration method [1] for complex eigenvalue computations. This code is able to take the toroidal and poloidal rotation of the plasma equilibrium into account for its linear wave and instability diagnostics. Test cases of this code are presented and compared with other existing codes, like CASTOR [3], ERATO, and PEST-1. Furthermore, recent and new results are presented with particular emphasis on MHD spectra for stationary tokamak equilibria. These equilibria include purely toroidal flow or toroidal and poloidal flow.

Physical model

The plasma inside a tokamak or accretion disk can be modeled by making use of the single-fluid MHD equations. These equations are

$$\rho\partial_t\mathbf{v} = -\rho\mathbf{v}\cdot\nabla\mathbf{v} - \nabla p + \mathbf{j}\times\mathbf{B} - \rho\nabla\Phi, \tag{1}$$

$$\partial_t p = -\mathbf{v}\cdot\nabla p - \gamma p\nabla\cdot\mathbf{v}, \tag{2}$$

$$\partial_t\mathbf{B} = -\nabla\times\mathbf{E}, \tag{3}$$

$$\partial_t\rho = -\nabla\cdot(\rho\mathbf{v}), \tag{4}$$

where the variables ρ, $\mathbf{v}$, p, and $\mathbf{B}$, Φ, and γ are the density, velocity, pressure, magnetic field, gravitational potential, and ratio of the specific heats, respectively. Here, the current density $\mathbf{j} = \nabla\times\mathbf{B}$. Furthermore, the simplified form of Ohm's law, $\mathbf{E} = -\mathbf{v}\times\mathbf{B} + \eta\mathbf{j}$, and the equation $\nabla\cdot\mathbf{B} = 0$ must be satisfied. Here, η is the resistivity. To relate the plasma pressure p with the temperature T we make use of the ideal gas law, $p = \rho T$.

We linearize the MHD equations (1)–(4) by assuming *time-dependent* fluctuations about a *time-independent* axisymmetric equilibrium. For the fluctuations we take the following dependence

$$f_1(\psi,\vartheta,\varphi,t) = \sum_m \hat{f}_{1,m}(\psi)\exp\left[\mathrm{i}(m\vartheta + n\varphi - \omega t)\right], \tag{5}$$

where ϑ is an angle chosen such that the magnetic field lines become straight in the (ϑ,φ)-plane. The resulting linearized MHD equations are discretized using a combination of quadratic and cubic Hermite elements in the ψ-direction and a Fourier representation in the poloidal direction. This results in a generalized eigenvalue problem:

$$\mathsf{A}x = \lambda\mathsf{B}x, \tag{6}$$

where $\lambda \equiv -\mathrm{i}\omega$ and $x = [\rho_1, v_{1,\psi}, v_{1,\vartheta}, v_{1,\varphi}, T_1, A_{1,\psi}, A_{1,\vartheta}, A_{1,\varphi}]^{\mathrm{T}}$. Here, the $A_{1,i}$'s are the components of the perturbed vector potential. As boundary condition we consider a perfect conducting wall.

Jacobi-Davidson method

The Jacobi-Davidson algorithm [1] has been used to find the eigenvalues of the generalized eigenvalue problem. The generalized eigenvalue problem of the linearized MHD equations (6) can be written as

$$\mathbf{Q}x = \mu x, \tag{7}$$

where $\mathbf{Q} = (\mathbf{A} - \sigma\mathbf{B})^{-1}\mathbf{B}$ and the inverse shifted eigenvalue $\mu = 1/(\lambda - \sigma)$. Here, σ is the specified target frequency. Due to the special block tridiagonal form of $\mathbf{A} - \sigma\mathbf{B}$, its $\mathbf{LU}$ decomposition can be computed easily.

At the k-th step of the iterative Jacobi-Davidson (JD) algorithm, an eigenvector x is approximated by a linear combination of search vectors $v_j (j =, 1, 2, \ldots, k; k \ll N)$. Consider the $N \times k$ matrix $\mathbf{V}_k$, whose columns are the search vectors v_j, then $x \approx \mathbf{V}_k s \equiv u$ for some k-vector s. The vectors v_j are made orthonormal to each other using the Modified Gram-Schmidt method.

Let θ denote the approximated eigenvalue associated with the approximated eigenvector u such that the residual vector $r = (\mathbf{Q} - \theta\mathbf{I})u$ is orthogonal to the k search directions.

In order to obtain a new search direction, the JD method requires the approximated solution of the correction equation [1]. This equation can be solved by some iterative method like GMRES [2].

Applications

In the PHOENIX code the eigenvalues are normalised to the Alfvén time, $\hat{\lambda} = R_\mathrm{M}\lambda/v_\mathrm{A}$, where R_M and v_A are the radius and the Alfvén speed on the magnetic axis, respectively.

As a first test we compare the PHOENIX code with other existing codes, like CASTOR [3], ERATO, and PEST-1, by investigating an isolated unstable global mode. For this test case, we used the analytical solution of the Grad-Shafranov equation [4], [5] given by Soloviev [6] for the equilibrium. A JET-like cross-section has been used which is given by $\varepsilon = 1/3$ and $E = 2$. Computing the numerical solution of this equilibrium and other equilibria presented in this paper we have used the code FINESSE [7]. The PHOENIX results are presented in Table 1, together with the results of the other codes. The results of the other codes are taken from Kerner et al. [3]. In Table 1, $q(0)$ and $q(1)$ are the safety factor at the magnetic axis and at the boundary, respectively. Furthermore, the eigenvalues are normalised to the poloidal Alfvén time and therefore the growth rates have been multiplied by the safety factor at the plasma boundary, $\tilde{\lambda} = q(1)\hat{\lambda} = q(1)R_\mathrm{M}\lambda/v_\mathrm{A}$. It is clear from this table that the agreement between the different codes is within 1%.

Table 1: Comparison of the eigenvalue $\tilde{\lambda}^2$ for a specific Soloviev equilibrium from different ideal MHD spectral codes. The toroidal mode number $n = -2$.

$q(0)$	$q(1)$	PHOENIX	CASTOR	KERNER	PEST-1	ERATO	Degtyarev	NOVA	Spector
0.3	0.523	0.431	0.429	0.413	0.427	0.431	0.430	0.430	0.432
0.7	1.220	0.120	0.120	0.118	0.119	0.120	0.121	0.119	0.118

For the second test case, we compare PHOENIX also with the codes CASTOR, MARS [8] and TERPSICHORE [9]. Again, we use the Soloviev solution for the equilibrium. For the cross-section we consider two elliptical, $E = 2$, and one circular, $E = 1$, cross-section together with an inverse aspect ratio $\varepsilon = 1/3$. The results of this case are presented in Table 2. The agreement between the different codes is typically within 0.5%.

As far as we know there is no simple published test case for an equilibrium with purely toroidal flow. Therefore we start with an equilibrium based on the Soloviev solution of the first static test case. In case the plasma rotates, the temperature is assumed to be a flux function. For

Table 2: Comparison of the eigenvalue $\hat{\lambda}$ for a specific Soloviev equilibrium with $\varepsilon = 1/3$ from different ideal MHD spectral codes.

n	q(0)	E	PHOENIX	CASTOR	ERATO	MARS	TERPSICHORE
-2	0.3	2	1.255	1.255	1.26	1.26	1.25
-2	0.7	2	0.284	0.284	0.284	0.284	0.284
-3	0.75	1	0.05397	0.05384	0.0541	0.0533	0.0538

the equilibrium the following flux functions have been used:

$$I^2(\psi) = A, \quad \rho_0(\psi) = 1, \quad p_0(\psi) = AB(1 - 0.9\psi), \tag{8}$$

where $B = 2.5$. The cross-section has been specified by ellipticity $E = 2$, trangularity $T = 0.2$, and rectangularity $Q = 0.01$ and the inverse aspect ratio $\varepsilon = 0.381966$. The safety factor on the magnetic axis $q(0) = 0.7$. Figure 1 shows the growth rate and oscillation frequency as function of the rotation frequency $\Omega(0)$ on the magnetic axis. Here, we used a toroidal mode number $n = -2$ and poloidal mode numbers $m = [-3, 7]$. It is clear from this figure that this particular unstable mode becomes less unstable if one includes toroidal flow, regardless of the direction of the flow. If the toroidal flow has some shear the mode becomes even more stable. This stabilizing effect has also been found by Chandra et al. [10] for classical and neoclassical tearing modes. The figure also shows that the oscillation frequency scales linearly with the rotation frequency on the magnetic axis ($\Omega = \text{cst.} \Rightarrow \text{Re}(\omega) \approx -1.98\Omega(0)$ and $\Omega = \Omega_0(1 - 0.9\psi) \Rightarrow \text{Re}(\omega) \approx -1.02\Omega(0)$).

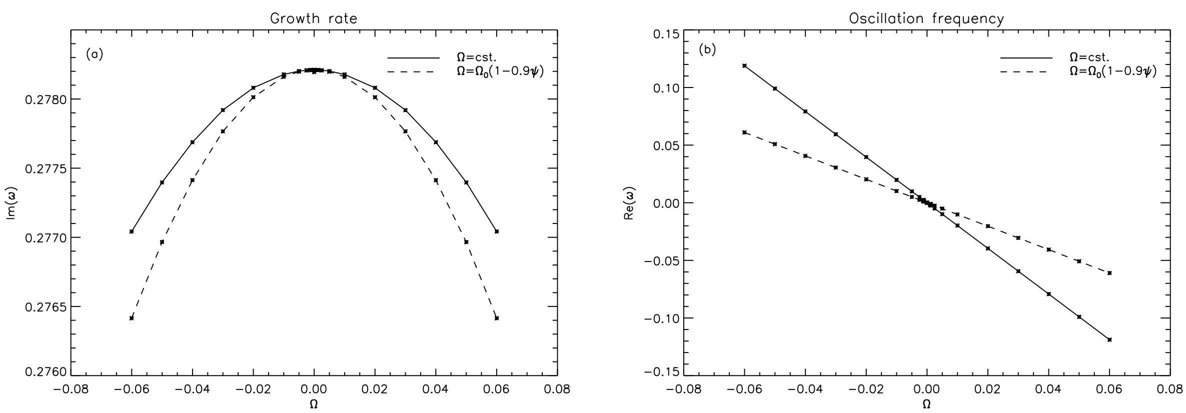

Figure 1: The growth rate (a) and oscillation frequency (b) for static equilibrium (solid) and an equilibrium with toroidal flow (dashed). The rotation frequency Ω on the x-axis is the value on the magnetic axis and has been normalized with respect to the Alfvén time on the magnetic axis.

A more stringent test is to find the damped, global Toroidal Flow induced Alfvén Eigenmode (TFAE) which was found by van der Holst et al. [11]. They use an equilibrium where the density is assumed to be a flux function. We use the following flux functions for the equilibrium:

$$\begin{aligned} I^2(\psi) &= A(1 - 0.0285\psi + 0.01045\psi^3), \quad & \rho(\psi) &= 1 - 0.85\psi, \\ p_0(\psi) &= AB(1 - 1.1\psi + 0.2\psi^2), & \Omega(\psi) &= C, \end{aligned} \tag{9}$$

where $B = 0.0217$ and $C = 0.0952$. These flux functions differ slightly from the ones used by van der Holst et al. [11]. Using the same strategy as described in that article we were able to find the TFAE, which has in our case $\text{Re}(\omega) = -0.197$. The η-convergence study of this

TFAE mode is shown in Figure 2(a), similar to [12]. This study shows that the TFAE damping rate $\text{Im}(\omega) \approx -1.4 \times 10^{-4}$. The perturbed normal velocity for three different poloidal mode numbers has been plotted in Figure 2(b). This shows that the $m = 2$ harmonic is the most dominant one. The near singular behaviour at $\sqrt{\psi} \approx 0.90$ is due to the interaction with the MHD continua.

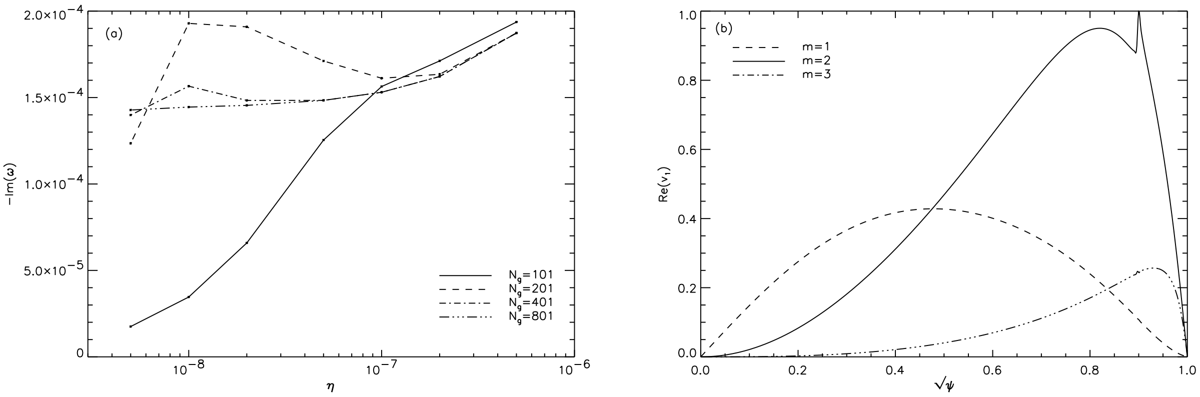

Figure 2: (a) Continuum damping of the flow-induced global gap mode, and (b) the normal component of the perturbed velocity; $\eta = 5 \times 10^{-8}$, $N_\text{g} = 801$, $m = [-1, 5]$.

Conclusions

The PHOENIX code has been compared with other existing codes. This comparison shows a good agreement between the different existing codes for ideal MHD spectrosopy of static equilibria. A simple test case with toroidal flow shows that an unstable mode becomes less unstable if there is toroidal rotation. Furthermore, if the rotation profile has some shear the stabilizing effect is stronger. In this test case the oscillation frequency scales linearly with the toroidal rotation on the magnetic axis.

The code was used to reproduce the Toroidal Flow induced Alfvén Eigenmode found by van der Holst et al. [11].

In the near future the PHOENIX code will be used to preform spectral studies of equilibria with purely toroidal and external gravity or equilibria with toroidal and poloidal flow.

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