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Analytical Solutions of dust acoustic Solitons

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We investigate a dusty plasma whose constituents are electrons, ions and heavy dust grains with a negative charge. The mass of those grains is much larger than the mass of the ions. To investigate this problem, it is customary [1] to reduce the set of equations to an ‘energy integral’ equation: a kinetic energy part together with potential energy (the Sagdeev potential). Since this equation cannot be solved analytically, one solves it in a numerical way. So only graphical solutions are available. Our aim is to study this basic model of dusty plasma in a different way. We opt for an analytical approach, although some restrictions will apply, to observe the importance of the different parameters involved. Moreover the associated analytical expressions may be used as initial conditions for a numerical approach.

The governing equations of the dust particles are (in normalized form):

$$\frac{\partial N_d}{\partial t} + \frac{\partial N_d u_d}{\partial x} = 0, \tag{1}$$

$$\frac{\partial u_d}{\partial t} + u_d \frac{\partial u_d}{\partial x} = \beta \frac{\partial \Psi}{\partial x} \qquad \text{with } \beta = Z_d \frac{m_i}{m_d}, \tag{2}$$

$$\frac{\partial^2 \Psi}{\partial x^2} = Z_d n_d + n_e - n_i. \tag{3}$$

Here N_d, u_d and $-eZ_d$ represent the normalized dust density, dust velocity and dust charge, respectively. The quantity Ψ represents the electrostatic potential normalized to the electron temperature; n_e and n_i the normalized densities of electrons and ions, respectively. These latter quantities are expressed as follows:

$$n_e = \delta e^{\Psi} \qquad \text{with } \delta = \frac{n_{e0}}{n_{i0}}, \tag{4}$$

$$n_i = e^{-\sigma\Psi} \qquad \text{with } \sigma = \frac{T_e}{T_i}. \tag{5}$$

The relevant quantities are then written as an equilibrium value together with their perturbed counterparts. Hence we have

$$N_d = \alpha + n_d \qquad \text{with } \alpha >> n_d, \tag{6}$$

$$\Psi = 0 + \Psi \qquad \text{with } \Psi \sim \mathscr{O}(n_d), \tag{7}$$

$$u_d = 0 + u_d \qquad \text{with } \Psi \sim \mathscr{O}(n_d). \tag{8}$$

Next, to find solutions to this set of Eqs. (1)-(3), it is customary to transform the set of (x,t) variables to the unique variable $\eta = c(x - Vt)$. One thus looks for stationary waves if they are present in this problem. Notice further that boundary conditions require that all perturbed quantities, as defined in (6)-(8), vanish for $\eta \to \pm\infty$.

Substitution of (4)-(8) into the set of equations (1)-(3), transforming it to that new variable η and finally integrating it where possible, gives

$$\begin{aligned} -Vn_d + \alpha u_d + n_d u_d &= 0, && (9)\\ -Vu_d + \frac{1}{2}u_d^2 - \beta\Psi &= 0, && (10)\\ c^2\frac{d^2\Psi}{d\eta^2} &= \alpha Z_d + Z_d n_d + \delta + \delta\Psi + \frac{1}{2}\delta\Psi^2 - 1 + \sigma\Psi - \frac{1}{2}\sigma^2\Psi^2, && (11)\end{aligned}$$

where $n_e = \delta e^{\Psi}$ and $n_i = e^{-\sigma\Psi}$ are developed in a Taylor series. Only the lowest order contributions are taken into account. Similar to the case of ordinary ion acoustic waves [2], we will reduce this set of equations to one single wave equation.

We first observe from (11) that in equilibrium

$$Z_d = \frac{1-\delta}{\alpha}, \qquad (12)$$

which expresses the quasi neutrality condition of this dusty plasma.

Next we derive the quantities n_d and u_d from Eqs. (9) and (10) in terms of Ψ. The nonlinear terms will be approximated by the solutions of the corresponding linearized equations. We get, after substitution of these results into (11), the nonlinear wave equation

$$c^2\frac{d^2\Psi}{d\eta^2} = \frac{3}{2}(1-\delta)\frac{\beta^2\Psi^2}{V^4} - \frac{(1-\delta)\beta}{V^2}\Psi + (\delta+\sigma)\Psi + \frac{1}{2}(\delta-\sigma^2)\Psi^2. \qquad (13)$$

This equation ressembles an integrated KdV equation. Hence a solitary-wave solution is expected. The following parameters must be determined: the amplitude and the velocity (note that in many cases the velocity is chosen as a free parameter; in [2] we have argued that our choice avoids secular terms using a perturbation approach) in terms of wave number c and other parameters involved. However, a huge task remains: how to find the velocity V of this solitary wave? To avoid this problem, we shall make use of the fact that the tail of a solitary wave moves with the same velocity as the bulk of the wave. We anticipate that a KdV solitary wave will be found and therefore, in the limit for $\eta \to +\infty$, the expected solution will behave like $\exp(-2\eta)$.

Substitution of this latter asymptotic value into (13) and neglecting terms $\sim e^{-4\eta}$, renders the following expressions for the velocity

$$V^2 = \frac{(1-\delta)\beta}{(\delta+\sigma)-4c^2} \qquad \text{or} \qquad V = \pm\sqrt{\frac{(1-\delta)\beta}{(\delta+\sigma)-4c^2}}. \tag{14}$$

Obviously (with the assumption that $\delta < 1$) only relative long wavelengths (small wave numbers c) are allowed. Remark that we deal with solitary waves which move from the right to the left and vice versa.

The final step is to determine the amplitude of the solitary wave. We may for instance use the tanh method [3]. So we introduce a new variable $Y = \tanh[c(x-Vt)] = \tanh(\eta)$ and express the solution as $A(1-Y^2)$, which is KdV like. After some algebra we get

$$\Psi = \frac{12c^2(1-\delta)}{(\delta^2+\sigma^2)+\delta(1+s^2)-3\left(4c^2-(\delta+\sigma)\right)^2}\,\mathrm{sech}^2[c(x-Vt)]. \tag{15}$$

The amplitude thus only depends on the parameters δ, σ and c.

Similar to our analysis in ref [2] one can set up a perturbation scheme to improve the obtained analytical results. The smallness parameter in this perturbation approach will be c^2. Note that our approach differs from the well known reductive perturbation theory since secular terms appear in the latter one (see ref [2] for more details). This work is still in progress.

The results obtained so far may be used as initial conditions for a numerical approach. Moreover existing numerical schemes can be checked upon their validity (within the limited range of parameters allowed). Next it seems that related problems (with variable dust charge for instance [4]) can be studied in the same analytical manner.

References

[1] P.K. Shukla and A.A. Mamun, Introduction to dusty Plasma Physics, IOP London (2002)

[2] W. Malfliet, J. Plasma Phys. **56**, 444 (1996)

[3] W. Malfliet, Am. J. Physics **60**, 650 (1992)

[4] S. Moolla, R. Bharauthram and S. Baboolal, Phys. Plasmas **12**, 042310 (2005)