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Dynamics on a Surface of Equilibrium

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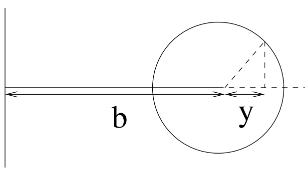
Considering the continuous interplay of charged particles with magnetic surfaces in fusion devices, it seems to be of interest to consider their dynamic behaviour once an equilibrium state has been reached. For an axisymmetric configuration in the large aspect ratio approximation, we show that for a flux surface of circular cross section, particles are constrained on a toroidal flux surface if their angular momentum lies between certain limits. A second requirement is that the particles have to rotate with a characteristic frequency which is a combination of the value of the flux surface and the energy and mass of the particles.

Consider a point at position



$$\mathbf{x} = (b+y)\cos\phi\ \mathbf{e}_1 + (b+y)\sin\phi\ \mathbf{e}_2 + z\,\mathbf{e}_3 \tag{1}$$

in the toroidal system

$$ds^2 = dy^2 + (b+y)^2 d\phi^2 + dz^2. \tag{2}$$

The magnetic field is defined to be of the form

$$\mathbf{B} = \frac{1}{b+y}\nabla\psi \times (0,1,0) + B_\phi(0,1,0), \tag{3}$$

which, in an axisymmetric system where $\partial/\partial\phi = 0$, becomes

$$\frac{\partial}{\partial z}\Big[(b+y)A_\phi - \psi\Big] = 0, \qquad \frac{\partial}{\partial y}\Big[(b+y)A_\phi - \psi\Big] = 0, \tag{4}$$

for the radial and axial components. Here A_ϕ is the azimuthal component of the vector potential. Multiplying (4) respectively by $\dot{z}$ and $\dot{y}$ and then adding, we obtain

$$\frac{d}{dt}\Big[(b+y)A_\phi - \psi\Big] = 0, \tag{5}$$

which is the requirement for the point to lie on an equilibrium surface. From this we obtain

$$(b+y)A_\phi - \psi = \psi_c, \tag{6}$$

where ψ_c is a constant, signifying a specific flux surface. From the Lagrangian [1]

$$\mathscr{L} = \frac{m}{2}\Big[\dot{y}^2 + (b+y)^2\dot{\phi}^2 + \dot{z}^2\Big] + \frac{e}{c}\Big[\dot{y}, (b+y)\dot{\phi}, \dot{z}\Big] \cdot \Big(A_y, A_\phi, A_z\Big) \tag{7}$$

we obtain the equation of motion for the axisymmetric component,

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathscr{L}}{\partial \phi}, \tag{8}$$

from which is follows that

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{\phi}} = 0, \qquad \frac{\partial \mathscr{L}}{\partial \dot{\phi}} \equiv \text{constant, say } J_\phi. \tag{9}$$

The energy is given by

$$E = \dot{q}\frac{\partial \mathscr{L}}{\partial \dot{q}} - \mathscr{L} = \frac{m}{2}\left(\dot{y}^2 + \dot{z}^2\right) + \frac{m}{2}(b+y)^2\dot{\phi}^2. \tag{10}$$

It follows that

$$m(b+y)^2\dot{\phi} + \frac{e}{c}(b+y)A_\phi = J_\phi. \tag{11}$$

Substituting (6) into this expression leads to

$$\dot{\phi} = \frac{M}{m(b+y)^2}\ , \tag{12}$$

where the definition

$$M = J_\phi - \frac{e}{c}(\psi - \psi_c) \tag{13}$$

was used. With this result the expression for the energy (11) becomes

$$E = \frac{m}{2}\left[\left\{\frac{d(b+y)}{dt}\right\}^2 + \dot{z}^2\right] + \frac{1}{2m}\frac{M^2}{(b+y)^2}\ , \tag{14}$$

where we have used the fact that $d(b+y)/dt = \dot{y}$. Let

$$r = b + y \tag{15}$$

so that

$$E = \frac{m}{2}\left[\dot{r}^2 + \dot{z}^2\right] + \frac{1}{2m}\frac{M^2}{r^2}\ . \tag{16}$$

From (5) we obtain

$$\dot{z} = -\dot{r}\frac{[rA_\phi - \psi]_{,y}}{[rA_\phi - \psi]_{,z}} \tag{17}$$

which, upon substitution into the energy (17) leads to

$$E = \frac{m}{2}\dot{r}^2\Big(1 + \Phi(r)\Big) + \frac{1}{2m}\frac{M^2}{r^2}\ , \tag{18}$$

where the definition

$$\Phi(r) = \left[\frac{(rA_\phi - \psi)_{,y}}{(rA_\phi - \psi)_{,z}}\right]^2 \tag{19}$$

was used. It follows that

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - \frac{1}{2m}\frac{M^2}{r^2}\right)\frac{1}{1+\Phi(r)}}}\,. \tag{20}$$

A readily tractable application is that of the invariant surface being a circle. Mapping (6) onto this surface, we can write

$$\psi - rA_\phi = \psi_0 \equiv r^2 + z^2 = (b+y)^2 + z^2 \tag{21}$$

up to a constant κ which takes care of the units. Here $\psi_0 = |\psi_c|$. It follows that

$$\frac{\partial}{\partial y}\Big[rA_\phi - \psi\Big] = 2(b+y) \qquad \text{and} \qquad \frac{\partial}{\partial z}\Big[rA_\phi - \psi\Big] = 2z, \tag{22}$$

so that (20) becomes

$$\Phi(r) = \frac{(b+y)^2}{z^2} = \frac{r^2}{\psi_0 - r^2} \tag{23}$$

and (21) is written as

$$dt = m\sqrt{\psi_0}\,\frac{rdr}{\sqrt{-2mEr^4 + (2mE\psi_0 + M^2)r^2 - M^2\psi_0}}. \tag{24}$$

Substituting $r^2 = \chi$, so that $2rdr = d\chi$, it follows that

$$dt = \frac{m}{2}\sqrt{\psi_0}\,\frac{d\chi}{\sqrt{-2mE\chi^2 + (2mE\psi_0 + M^2)\chi - M^2\psi_0}}. \tag{25}$$

From Gradshteyn and Ryzhik (p.81, expression 2.261) we obtain the result

$$\int dt = t = -\sqrt{\frac{m\psi_0}{8E}}\sin^{-1}\frac{-4mE\chi + M^2 + 2mE\psi_0}{M^2 - 2mE\psi_0}, \tag{26}$$

which is rewritten as

$$(b+y)^2 = \frac{1}{2}\left(\frac{M^2}{2mE} + \psi_0\right) + \frac{1}{2}\left(\frac{M^2}{2mE} - \psi_0\right)\sin(\Omega_0 t) \tag{27}$$

where

$$\Omega_0 = \sqrt{\frac{8E}{m\psi_0}} \tag{28}$$

is the characteristic frequency of a particle with mass m and energy E. From (22) we obtain

$$z^2 = \psi_0 - (b+y)^2 = \frac{1}{2}\left(\psi_0 - \frac{M^2}{2mE}\right)\Big[1 + \sin(\Omega_0 t)\Big]. \tag{29}$$

For z to be a real function, the condition

$$\psi_0 - \frac{M^2}{2mE} > 0 \tag{30}$$

must be satisfied, or equivalently,

$$(M+\sqrt{2mE\psi_0})(M-\sqrt{2mE\psi_0})<0. \tag{31}$$

This gives the necessary condition for a particle to get trapped on a flux surface. We can rewrite these expressions in a different form by defining

$$A=\frac{1}{2}\left(\psi_0+\frac{M^2}{2mE}\right) \qquad \text{and} \quad B=\frac{1}{2}\left(\psi_0-\frac{M^2}{2mE}\right). \tag{32}$$

Equations (28) and (30) become respectively

$$(b+y)^2 = A-B\sin(\Omega_0 t) \tag{33}$$

$$\text{and} \quad z^2 = B\Big[1+\sin(\Omega_0 t)\Big]. \tag{34}$$

Using (35), expression (13) can be rewritten as

$$\frac{d\phi}{dt}=\frac{M}{m}\frac{1}{A-B\sin(\Omega_0 t)}, \tag{35}$$

so that

$$\phi=\frac{M}{m}\int\frac{dt}{A-B\sin(\Omega_0 t)}. \tag{36}$$

Since

$$A^2-B^2=\frac{M^2\psi_0}{2mE} \tag{37}$$

it follows that $A^2>B^2$, so that we can use Gradshteyn and Ryzhik (p.174, expression 2.551.3) to obtain

$$\phi=\frac{2M}{m\Omega_0}\frac{1}{\sqrt{A^2-B^2}}\tan^{-2}\left[\frac{A\tan\left(\frac{1}{2}\Omega_0 t\right)-B}{\sqrt{A^2-B^2}}\right]. \tag{38}$$

With observation (39) and also

$$\frac{2M}{m\Omega_0}\frac{1}{\sqrt{A^2-B^2}}=1, \tag{39}$$

it follows that

$$\tan^2\phi=\frac{A}{\sqrt{A^2-B^2}}\left[\tan\left(\frac{\Omega_0 t}{2}\right)-\frac{B}{A}\right]. \tag{40}$$

Thus expression (36) and (42) give the complete analytical solution for charged particles trapped on isoflux surfaces in the low-aspect ratio approximation. In this simplified model it is seen that the particle rotates with a characteristic frequency Ω_0.

[1] L D Landau, E M Lifshitz, 1975, The Classical Theory of Fields, 4th English revised ed., (Pergamon Press, Oxford) p.45, eq.(16.4)

[2] I S Gradshteyn, I M Ryzhik, 1980, Tables of Integrals, Series and Products (Academic Press, New York)