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Shaping effects on the ideal (1, 1) mode in low-shear tokamaks

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Introduction In tokamak plasmas in the "hybrid scenario", q is slightly larger than 1 in a wide area in the plasma core, and increases up to $q_a \sim$ 3-5 in the edge region [1]. It is well-known that such equilibria are susceptible to a pressure-driven, internal (1, 1) ideal MHD instability with an eigenfunction of "quasi-interchange" (QI) character [2]. Stability of this mode requires that the value of q in the low-shear region is not too close to unity. Since the ideal (1, 1) mode in high-shear plasmas is very sensitive to the shape of the cross section [3], it is possible that shaping effects are of importance also for the QI mode. The present work investigates the leading-order effects of a non-circular cross section on the QI mode using the equations derived in [3] for analysing the ideal (1, 1) mode in weakly non-circular, toroidal equilibria with a general q-profile.

Equilibrium We consider a toroidal plasma equilibrium valid up to order ε^2 in the inverse aspect ratio $\varepsilon = r/R_0$. The Shafranov shift Δ of such a plasma is given by $d\Delta/dr = -r(\beta_p + l_i/2)/R_0$, where the inductance l_i and the poloidal beta value $\beta_p \sim 1$ are defined as usual [3]. In addition to the toroidal shift of the flux surfaces, we assume that they are also slightly elliptical and triangular: $\rho(r, \theta) = r + \tilde{E}(r)\cos 2\theta + \tilde{T}(r)\cos 3\theta$, where θ is the poloidal angle. A self-consistent equilibrium theory including the weak ellipticiy and triangularity above can be found in [3]. In order to simplify the stability calculation we assume that $q = 1 - \Delta q = q_0 = const$, and $|\Delta q| << 1$, for $0 \le r \le r_1$, whereas for $r_1 \le r \le a$ we assume that $q(r)$ increases from q_0 at $r = r_1$ up to a value q_a at the plasma edge $r = a$. We assume further that the pressure profile is parabolic. Such a pressure profile together with $q \approx 1$ leads to $\beta_p \approx const$ in the low-shear region. Furthermore, it turns out that $q = const$ gives $\tilde{E}(r) = \tilde{E}_0 r$ and $\tilde{T}(r) = \tilde{T}_0 r^2$ in the region $0 \le r \le r_1$. In terms of the usual shaping parameters κ and δ this gives $e = (\kappa - 1)/2 = -\tilde{E}/r = -\tilde{E}_0 = const$ and $\delta = 4\tilde{T}/r = 4\tilde{T}_0 r$ in the low-shear region. In the examples shown in the next section we use the notation κ_1 for the elongation of the flux surfaces in the *whole* inner region, and δ_1 for the triangularity at the edge $r = r_1$ of the low-shear region.

Stability In addition to the toroidal effect of order ε^2, the leading-order shaping effects, of order $e\varepsilon^2$ and the $e\varepsilon\delta$, on the growth rate of the QI mode can be obtained from the stability equations in [3]. Furthermore, the quasi-cylindrical contributions of order e^2 and δ^2 are derived in [4]. The stability problem consists of a system of equations where the $m = 1$ amplitude ξ_1 couples to a set of side-band modes ξ_m, where $m = -2, -1, 0, 2, 3, 4$. For the simplified, but still reasonably realistic equilibrium defined above it turns out that i) all side-band equations can be solved analytically, and ii) the final equation for ξ_1 takes the simple form $(r^3 Q\xi_1')' + \lambda r^3 \xi_1 + \sigma r^3 = 0$, where $Q = (\Delta q)^2 + 3\gamma^2/\omega_A^2$ and λ and σ are constants given by [4]

$$\lambda = \lambda^{(\delta^2)} + \lambda^{(e\varepsilon^2)} + \lambda^{(e\varepsilon\delta)} = -\frac{3\Delta q \delta_1^2}{r_1^2} + \frac{3\beta_p(\kappa_1 - 1)}{R_0^2}\left(1 - \frac{2\delta_1}{\varepsilon_1}\right)$$

$$\sigma = \sigma^{(e^2)} + \sigma^{(\varepsilon^2)} + \sigma^{(e\varepsilon^2)} = \frac{24(\Delta q)^2(\kappa_1 - 1)^2 F_3}{r_1^6} + \frac{16\beta_p^2(3 - 2\kappa_1)F_2}{r_1^4 R_0^2}$$

Boundary conditions are $\xi_1'(0) = \xi_1(r_1) = 0$, and a normalization condition $\int r^3 \xi_1 dr = 1$, where the integration is from $r = 0$ to $r = r_1$, has been used. The solution of the equation for ξ_1 satisfying the boundary conditions can be written in terms of the Bessel function J_1 as $\xi_1(r) = (\sigma/\lambda)[r_1 J_1(kr)/rJ_1(kr_1) - 1]$, where $k \equiv (\lambda/Q)^{1/2}$. From the integral condition above one then obtains the eigenvalue equation in the form $D \equiv y[J_2(x)/xJ_1(x) - 1/4] - 1 = 0$, where $x \equiv kr_1$ and $y \equiv \sigma r_1^4/\lambda$. The factors $F_{2,3}$ in the expression for σ have to do with the side-bands $m = 2, 3$ in the edge region, and are defined as

$$F_m = \frac{m + 1 + A_m^+}{m - 1 - A_m^+} \quad \text{where} \quad A_m^+ = \frac{r}{\xi_m}\frac{d\xi_m}{dr}\bigg|_{r=r_1+0}$$

Fig. 1 shows D as a function of $(\gamma/\omega_A)^2$ for a plasma with a quadratic q-profile in the edge region (and $q_a = 4$) (this q-profile, and $\varepsilon_a = 0.3$, has been used in all the figures) and $r_1 = 0.5a$, $\beta_p = 0.3$, $\kappa_1 = 1.3$, $\delta_1 = 0.0$ and $q_0 = 1.01$. It is seen that there is an infinite number of solutions to the eigenvalue equation $D = 0$, and therefore an infinite number of modes. The spectrum accumulates at the frequency $-(\gamma/\omega_A)^2 = (\omega/\omega_A)^2 = (q_0 - 1)^2/3$ as $x \to \infty$. The number of nodes in the eigenfunction increases with decreasing $(\gamma/\omega_A)^2$, as shown in Fig. 2. The most unstable mode can be identified as the QI mode. Various ways of approximating the eigenvalues of these modes are discussed in [4]. A useful and fairly accurate expression for the growth rate of the QI mode is given by

$$\frac{\gamma^2}{\omega_A^2} = -\frac{(\Delta q)^2}{3} - \frac{\Delta q \delta_1^2}{16} + \frac{(\Delta q)^2(\kappa_1 - 1)^2 F_3}{12} + \frac{\beta_p(\kappa_1 - 1)\varepsilon_1^2}{16}\left(1 - \frac{2\delta_1}{\varepsilon_1}\right) + \frac{\beta_p^2 \varepsilon_1^2(3 - 2\kappa_1)F_2}{18}$$

If we set $\kappa_1 = 1$ and $\delta_1 = 0$ in this expression we recover the growth rate of the QI mode in a plasma with a circular cross section [2]. It turns out that the quasi-cylindrical shaping terms are negligibly small for typical hybrid-scenario plasmas due to the small value of $|\Delta q|$. So, the shaping effects are mainly determined by the competition between the destabilising (for $\delta_1/\varepsilon_1 < 0.5$) "Mercier" term and the last term where vertical elongation $\kappa_1 > 1$ is seen to improve the stability as compared with the case $\kappa_1 = 1$. The magnitude of the edge factor F_2 is apparently very important for the magnitude of this effect. For a quadratic q-profile with $q_a = 4$, F_2 can be approximated by $5.5(r_1/a)^2$ for

r_1/a in the range 0.2-0.5 [4]. Using this approximation, a rough criterion for when elongation becomes stabilising is given by $z > 0$, where $z \approx 5\beta_p(r_1/a)^2 + \delta_1/\varepsilon_1 - 0.5$. Thus, if the product of pressure and width of the low-shear region is sufficiently large, vertical elongation improves the stability of the ideal (1, 1) mode as compared with a plasma with circular cross section. The order of magnitude of the critical value of $\beta_p(r_1/a)^2$ is around 0.1 (when the triangularity is zero, or small). Fig. 3 shows the growth rate for different κ_1 as a function of β_p for a plasma with $r_1/a = 0.5$, $q_0 = 1.04$ and $\delta_1/\varepsilon_1 = 0.1$. The transition from destabilising to a significant stabilising effect of vertical elongation around $\beta_p \approx 0.4$ can be seen in this figure. Similar curves are shown also in Fig. 4, and here the stabilising effect of $|\Delta q|$, or the first term in the growth rate, is also shown. This figure is qualitatively similar to Fig. 5 in Ref. 5, a figure that shows the growth rate of the ideal $n = 1$ kink mode in JET computed with the MISHKA code. Fig. 5 in the present paper illustrates the stabilising effect of (positive) triangularity and the transition (around $\delta_1/\varepsilon_1 \approx -0.1$) where vertical elongation starts to become a stabilising instead of a destabilising effect. In Fig. 6, finally, the range of stability of the QI mode is shown in terms of the critical value (η_c) of a stability parameter η as a function of κ_1 for different q_0. The parameter η is defined as $\eta = \beta_p(r_1/a)^2\varepsilon_a/|q_0 - 1|$, and is obtained from the condition of marginal stability of a plasma with circular cross section together with the approximation $F_2 \approx 5.5(r_1/a)^2$. It is seen that (1, 1) stability in such a plasma is obtained when $\eta < \eta_c \approx 1.1$, more or less independent of the separate values of β_p, ε_a etc. Thus, increasing $\beta_p(r_1/a)^2$ requires a larger q_0 (for a fixed ε_a and $q_0 > 1$) for stability. However, increasing $\beta_p(r_1/a)^2$ also implies a better stabilising effect from elongation, as shown above, and therefore the stabilising effect of elongation improves with the value of q_0, which is what is shown in the figure. The critical q_0 when elongation becomes stabilising can be found by combining the condition $z \approx 0$ with $\eta \approx \eta_c$ (using $\eta_c \approx 1$ for simplicity): $q_{0,crit} \approx 1 + (0.1 - 0.2\delta_1/\varepsilon_1)\varepsilon_a = (\varepsilon_a = 0.3) = 1.03 - 0.06\delta_1/\varepsilon_1$. For a plasma with zero triangularity this gives $q_{0,crit} \approx 1.03$, which is approximately the value of q_0 when elongation starts to become stabilising in Fig. 6.

References

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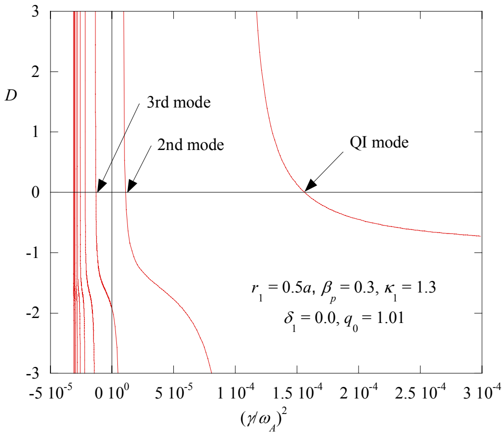

Fig. 1. Illustration of the eigenvalue equation D = 0 for the (1, 1) mode in a low-shear plasma with elongated cross section.

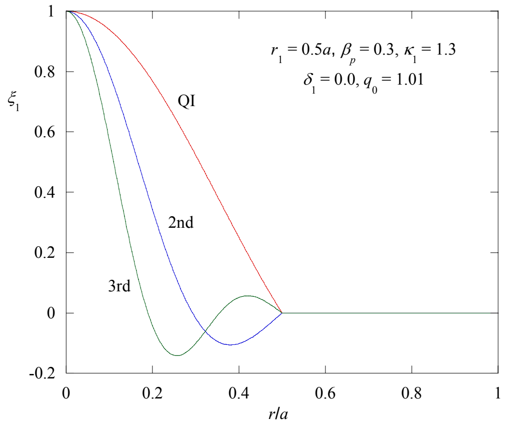

Fig. 2. The eigenfunctions for the three modes with largest growth rates in Fig. 1.

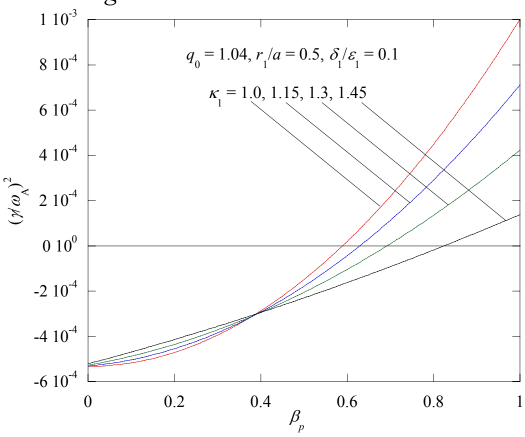

Fig. 3. Growth rate as a function of plasma pressure for different elongations. The figure illustrates the destabilising effect of β_p and the transition from destabilising to stabilising effect of elongation at a critical value of β_p.

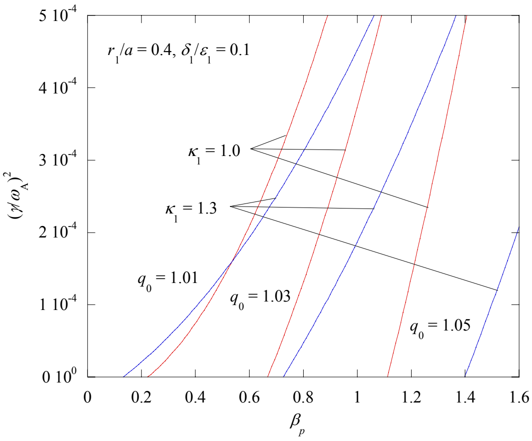

Fig. 4. Growth rate as a function of plasma pressure for different elongations and q_0. At low pressure (and small q_0) vertical elongation destabilises the QI mode, whereas elongation improves the stability at high pressure.

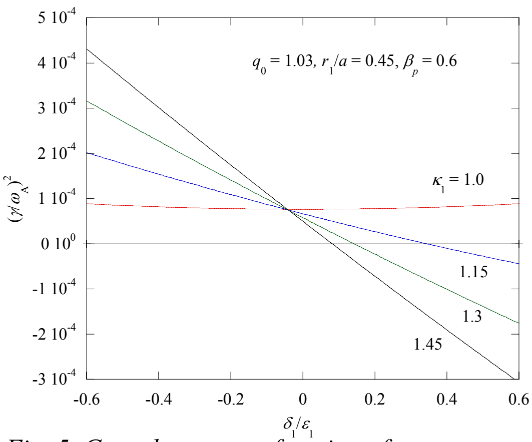

Fig. 5. Growth rate as a function of triangularity for different elongations. The figure illustrates the stabilising effect of triangularity as well as the transition from destabilising to stabilising effect of elongation at a critical value of triangularity

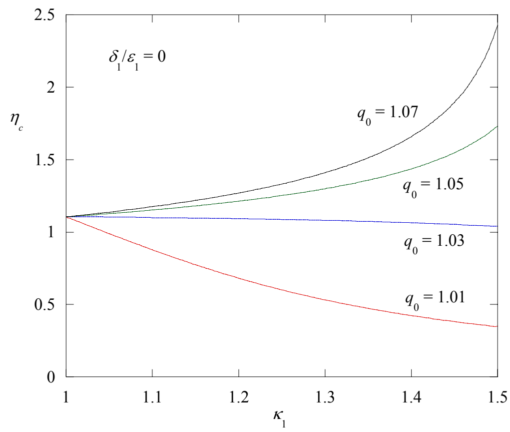

Fig. 6. The range of stability ($\eta < \eta_c$) expressed in terms of the parameter $\eta = \beta_p(r_1/a)^2\varepsilon_a/|q_0 - 1|$. When q_0 is sufficiently large, vertical elongation widens the range of stability of the QI mode.