*33rd EPS Conference on Plasma Phys. Rome, 19 - 23 June 2006 ECA Vol.**30I**, P-1.163 (2006)*

Geometrical properties of plasma equilibrium near fixed magnetic surface

A.A. Skovoroda, A.V. Zvonkov

RRC Kurchatov Institute, Moscow, Russia

In the paper some properties of plasma equilibrium near the fixed magnetic surface are analyzed. The knowledge of these properties can be useful for the choice of plasma boundary for 3D modeling and for consideration of constrains on plasma equilibrium. Two issues are considered – the possibility of magnetic field value prescription for specification of the local equilibrium and possible constrains on pressure profile coefficient imposed by surface shape.

1. Specification of the local equilibrium by magnetic field value prescription

It is known that for the specification of plasma equilibrium near the fixed magnetic surface it is necessary to prescribe not only the shape of a surface $\mathbf{r} = \mathbf{r}_r(\theta, \zeta)$ and two profile coefficients p', F' (or J'), but also an additional function of the angle variables θ, ζ [1]. As an additional function the "distance" to adjacent surface $\delta(\theta, \zeta) = |\nabla\Phi|^{-1}$ can be taken [1]. In [2] it was shown that the prescription of the surface in magnetic coordinates $\mathbf{r} = \mathbf{r}_r(\theta_m, \zeta)$ is sufficient also, in fact it means that the function $\eta(\theta, \zeta)$ that relates the magnetic poloidal angle with an arbitrary one is given, $\theta_m = \theta + \eta(\theta, \zeta)$. In [1] the possibility of the periodic potential $\varphi(\theta, \zeta)$ prescription is discussed also. We study here the possibility to use the value of magnetic field $B(\theta, \zeta)$ on the surface as an additional function.

The analysis near the fixed surface means that the shape of a reference surface and hence basic vectors $\mathbf{e}_2 = \dfrac{\partial \mathbf{r}}{\partial \theta}$, $\mathbf{e}_3 = \dfrac{\partial \mathbf{r}}{\partial \zeta}$, the metrics components g_{22}, g_{23}, g_{33} and normal vector $\mathbf{n} = \nabla\Phi / |\nabla\Phi| = [\mathbf{e}_2 \times \mathbf{e}_3] \big/ \sqrt{g_{22} g_{33} - g_{23}^2}$ are known.

The equilibrium equation entails the equation $\mathrm{div}[\mathrm{B} \times \nabla\Phi] = 0$, hence the vector $[\mathbf{B} \times \nabla\Phi]$ can be represented similarly to $\mathbf{B}$ [3]

$$2\pi\mathbf{B} = \left(1 + \frac{\partial \eta}{\partial \theta}\right)\frac{\mathbf{e}_3}{\sqrt{g}} + \left(\mu - \frac{\partial \eta}{\partial \zeta}\right)\frac{\mathbf{e}_2}{\sqrt{g}}; \qquad (1)$$

$$2\pi[\mathbf{B}\times\nabla\Phi] = -\left(J+\frac{\partial\varphi}{\partial\theta}\right)\frac{\mathbf{e}_3}{\sqrt{g}} + \left(F+\frac{\partial\varphi}{\partial\zeta}\right)\frac{\mathbf{e}_2}{\sqrt{g}}\,. \qquad (2)$$

The square of the latter expression gives the equation for φ, where all terms and coefficients are known,

$$4\pi^2B^2\left(g_{22}g_{33}-g_{23}^2\right) = \left(J+\frac{\partial\varphi}{\partial\theta}\right)^2 g_{33} - 2\left(J+\frac{\partial\varphi}{\partial\theta}\right)\left(F+\frac{\partial\varphi}{\partial\zeta}\right)g_{23} + \left(F+\frac{\partial\varphi}{\partial\zeta}\right)^2 g_{22}\,. \quad (3)$$

The periodic solution of (3) should be found, the constants J, F being used to satisfy the conditions of periodicity and zero average $<\varphi>=0$.

This equation can be considered as a Hamilton-Jacobi equation and can be solved by the method of characteristics. Introducing a momentum **p** with components $p_2 = J + \partial\varphi/\partial\theta$, $p_3 = F + \partial\varphi/\partial\zeta$, the equation (3) is expressed as

$$H = 1/2\sum g_S{}^{ij} p_i p_j - 2\pi^2B^2 = |\mathbf{p}|^2/2 - 2\pi^2B^2 = 0,$$

here $g_S{}^{33} = g_{22}/(g_{22}g_{33}-g_{23}^2)$, $g_S{}^{22} = g_{33}/(g_{22}g_{33}-g_{23}^2)$, $g_S{}^{23} = -g_{23}/(g_{22}g_{33}-g_{23}^2)$. This is the Hamiltonian of a particle with a unit mass and zero energy moving on a toroidal surface with metrics g_{ij}, $i,j=2,3$, in a potential $U=-2\pi^2B^2$. It is evident that the characteristics exist throughout the surface. The normalization of the surface metrics by $2\pi^2B^2$ shows that the characteristics are the geodesic lines of a surface with the new metrics. The freedom in characteristics direction (or p_2, p_3) allows to think that the periodic solution for φ exists, though practical realization of a solution is difficult.

The other functions and constants can also be found in this way [1, 4]. For instance, the scalar product of (1), (2) gives the equation for η and the vector product gives $|\nabla\Phi|$.

Let us consider as a simple example the axially symmetric equilibrium $B=B(\theta), \varphi=\varphi(\theta)$ on a torus with the major and minor radii R, a. If θ is the poloidal angle counted off from the inside equator and ζ is the toroidal angle, the metrics components are as follows: $g_{22}=a^2, g_{23}=0, g_{33}=a^2(A-\cos\theta)^2$, $A=R/a>1$. In dimensionless variables (J,F,φ are scaled by $2\pi B_0 a$) the equation (3) has the form $(J+\varphi')^2 = b^2 - F^2/(A-\cos\theta)^2$, where $b=B/B_0$. For solution $\varphi = -J\theta + \int\sqrt{b^2-F^2/(A-\cos\theta)^2}\,d\theta$ the relation $J = 1/2\pi\int_0^{2\pi}\sqrt{b^2-F^2/(A-\cos\theta)^2}\,d\theta$ is valid, and solution needs $F<F_c(A)$ and $J\neq 0$. The distance to adjacent surface is

$\delta = \delta_0 / \sqrt{b^2(A - \cos\theta)^2 - F^2}$. For instance, if the field value is constant ($b = 1$), the distance is minimal at outer side of a torus, and decrease of the toroidal field is compensated by increase of the poloidal field.

The example shows the well-known result that axially symmetric equilibrium on a torus is possible only if J is not zero. The configurations with special features, for example, the currentless systems or configurations with closed field lines restrict the choice of B .

2. The restrictions on pressure profile coefficient imposed by surface shape

Now consider the situation when the magnetic surface is fixed, but the equilibrium near it has some freedom. Let us consider the question if the surface shape imposes any restrictions on possible local equilibrium, for instance, on the pressure profile coefficient p' .

It is known that for any line on a surface the following relationship fulfills

$$\kappa_n^2 + \tau_n^2 - 2H\kappa_n + K = 0 , \tag{4}$$

here H , K are the properties of a surface - average and Gaussian curvature. The other values are the properties of a line - κ_n is normal curvature and τ_n is "normal torsion" of a line. They are defined as follows:

$$\boldsymbol{\kappa} = (\mathbf{b}\cdot\nabla)\mathbf{b} = \kappa_n\mathbf{n} + \kappa_g[\mathbf{b}\times\mathbf{n}],$$

$$(\mathbf{b}\cdot\nabla)\mathbf{n} = -\kappa_n\mathbf{b} + \tau_n[\mathbf{b}\times\mathbf{n}], \quad (\mathbf{b}\cdot\nabla)[\mathbf{b}\times\mathbf{n}] = -\kappa_g\mathbf{b} - \tau_n\mathbf{n} ,$$

here κ_g is the geodesic curvature. H and K depend on the point of a surface only, whereas κ_n and τ_n – also on the direction of a line on a surface.

If the surface is magnetic surface and the line is magnetic field line, from the equilibrium and Maxwell equations it follows that

$$\kappa_n = \mathbf{n}\cdot\nabla\left(p + \frac{B^2}{2}\right)/B^2 ,$$

$$\tau_n^2 = \frac{1}{4}\cdot\left([\mathbf{n}\times\mathbf{b}]\cdot \mathrm{rot}[\mathbf{n}\times\mathbf{b}] - \mathbf{b}\cdot \mathrm{rot}\mathbf{b}\right)^2 = \frac{1}{4}(s - j_{\|}/B)^2 ,$$

here s is local shear, $j_{\|}$ is parallel current. Resolving quadratic (4) we find

$$\kappa_n = H \pm \sqrt{H^2 - K - \tau_n^2} . \tag{5}$$

The choice of a sign here determines the direction of the field line: "minus" means that the curvature κ_n is large (H is usually negative, so both terms have the same sign) and the field line has predominantly poloidal direction, "plus" means smaller curvature and predominantly toroidal direction of the field. The combination $H^2 - K$ is always positive for a toroidal surface and the large aspect ratio results in the increase of the combination.

If the equilibrium is known, the relationship (5) is fulfilled identically. In this way (5) can be used as a check for 3D solvers. On the other hand, when the shape of a surface is prescribed, the curvatures H and K are known. At the same time both κ_n and τ_n depend on the pressure profile coefficient p'. So, the relationship (5) may impose restrictions on the pressure profile coefficient p', because the expression under the square root should be positive, and the value of the left side of (5) is also limited. But it is difficult to reveal the restrictions explicitly because the dependence on p' of some quantities in the expressions for κ_n and τ_n is not evident.

Some simple observations can be made nevertheless. Thus, the large pressure profile coefficient can be obtained if both terms on the right side of (5) have the same (negative) sign, what takes place for poloidal direction of field lines. In case of toroidal direction of field lines two terms have different signs and the largest terms $\sim 1/a$ cancel if aspect ratio is large. So, greater pressure coefficient may be expected for small aspect configurations.

Conclusions

The local plasma equilibrium can be specified if the surface and magnetic field value on it are prescribed as well as two profile coefficients. This approach can be useful for the choice of plasma boundary with desirable field value properties. The shape of the surface can constrain possible plasma pressure coefficient.

References

[1] Boozer A 2002 *Phys. Plasmas* **9** 3762

[2] Hegna C C 2000 *Phys. Plasmas* **7** 3921

[3] Pustovitov V D and Shafranov V D 1990 *Reviews of Plasma Physics Vol.15* (Consultants Bureau, New York) p 163

[4] Skovoroda A A 2006 *Plasma Physics Reports* **32** No11 *to be published*