Self-similar solutions of unsteady ablation flow

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The stability of ablative flows is of importance in inertial confinement fusion (ICF). Here we exhibit a family of exact self-similar solutions of gas dynamics equations with nonlinear heat conduction for semi-infinite slabs of perfect gases. Such self-similar solutions arise for particular but realistic time-increasing boundary pressure and heat flux laws, and are representative of the early stage of an ICF capsule irradiation, where a shock-wave front propagates upstream of an ablation front. Linear stability analyses of such time dependent solutions are performed by solving an initial and boundary value problem for linear perturbations [5]. Our goal here is to study these self-similar ablation flows, which are computed using a highly accurate numerical method, namely a dynamical multidomain Chebyshev spectral method [4, 7].

Self-similar mean flow

For one-dimensional motions in the x-direction of an inviscid heat-conducting fluid, obeying a perfect gas equation of state

\[ p = \rho RT, \quad \epsilon = TR/(\gamma - 1), \]

(1)

where \( \rho \) is the fluid density, \( p \) the fluid pressure, \( \epsilon \) the internal energy, \( T \) the fluid temperature, \( R \) the gas constant and \( \gamma \) the fluid adiabatic exponent, the equations of motion, written in Lagrangian form, are

\[
\begin{align*}
\partial_t (1/\rho) - \partial_m v_x &= 0, \\
\partial_t v_x + \partial_m p &= 0, \\
\partial_t (v_x^2/2 + \epsilon) + \partial_m (pv_x + \phi_x) &= 0,
\end{align*}
\]

(2)

where \( m \) is the Lagrangian coordinate (\( dm = \rho dx \)), \( v_x \) the velocity and \( \phi_x \) the heat flux, of expression

\[ \phi_x = -\kappa \partial_x T = -\kappa \rho \partial_m T = -\chi \left( \frac{\rho}{\rho_i} \right)^{-\mu} \left( \frac{T}{T_*} \right)^\nu \rho \partial_m T, \]

(3)

and \( \chi, \mu, \nu \) are fluid constants to be chosen such that \( \chi \geq 0, \mu \geq 0 \) and \( \nu \neq 1 \), \( \rho_i \) and \( T_* \) are characteristic density and temperature of the flow.

Solutions of self-similar type for system (2) have been investigated by several authors [8, 1, 3, 11, 12]. Here we consider time-dependent pressure and heat-flux boundary conditions [10]. At \( t = 0 \), the fluid of uniform density \( \rho_i \) is assumed to occupy the half-space \( m \geq 0 \), while a heat flux starts being applied along the plane \( m = 0 \). By choosing initial conditions to be

\[ \rho = \rho_i, \quad v_x = 0, \quad T = 0, \quad \text{for } m \geq 0, \]

(4)
and boundary conditions of the form
\[ p = p_\ast \left(\frac{t}{t_\ast}\right)^{2(\alpha - 1)}, \quad \varphi_x = \varphi_\ast \left(\frac{t}{t_\ast}\right)^{3(\alpha - 1)}, \quad \text{for } m = 0, \] (5)

system (2) admits a self-similar formulation. Here \( p_\ast, \varphi_\ast \) and \( t_\ast \) are characteristic pressure, heat flux and time. For convenience, we choose a dimensionless formulation of the equations based on the quantities \( \rho_i, R, \chi, t_\ast \). Applying the \( \Pi \)-theorem \[2\], the seven parameters \( \rho_i, R, \gamma, \chi, t_\ast, p_\ast, \varphi_\ast \) lead us to retain the three dimensionless numbers:
\[ \gamma, \quad B_p = p_\ast t_\ast R / \chi, \quad B_\varphi = \sqrt{\rho_i (t_\ast R / \chi)}^3. \] (6)

Henceforth, all the quantities are replaced by their dimensionless equivalents, while keeping the same notations. Introducing the self-similar variable \( \xi = m / t^\alpha \), with \( \alpha = (2\nu - 1) / (2\nu - 2) \), and time power-law dependencies for the physical variables
\[ \begin{align*}
\rho &= G(\xi), \\
v_x &= t^{\alpha - 1} V(\xi), \\
T &= t^{2\alpha - 1} \Theta(\xi), \\
\varphi_x &= t^{3(\alpha - 1)} \Phi(\xi),
\end{align*} \] (7)

the dimensionless system obtained from Eqs. (1,2,3) reduces to a system of ODEs:
\[ \frac{d}{d\xi} \begin{pmatrix} G \\ V \\ \Theta \\ \Phi \end{pmatrix} = \begin{pmatrix} G^2 N / D \\
\alpha \xi N / D \\
\alpha \xi F - 2(\alpha - 1) \Theta / (\gamma - 1) - \alpha \xi G \Theta N / D \end{pmatrix} \] (8)

with:
\[ N = (\alpha - 1)V + GF, \quad D = \alpha^2 \xi^2 - G^2 \Theta, \quad F = -\Phi G^{\mu - 1} \Theta^{-\nu}. \] (9)

The corresponding dimensionless form of equations (4) and (5) are
\[ G(\xi \to +\infty) = 1, \quad V(\xi \to +\infty) = 0, \quad \Theta(\xi \to +\infty) = 0, \]
\[ (G \Theta)(\xi = 0) = B_p, \quad \Phi(\xi = 0) = B_{\varphi}. \] (10)

Any solution of (8) satisfying Eqs. (10,11) necessarily includes the singularity \( D = 0 \). This singularity corresponds to an isothermal characteristic curve, say \( m / t^\alpha = \xi_s \), of the \((m,t)\)-plane, which is circumvented by introducing, as part of the solution, an isothermal shock wave at \( \xi = \xi_s \). Henceforth the boundary conditions (10) are replaced by the Rankine-Hugoniot conditions, at \( \xi = \xi_s \), for a non-isothermal shock wave with uniform upstream state given by (10), thus defining, along with Eq. (11), a nonlinear eigenvalue problem for system (8).
The highly accurate numerical method we have devised [7] consists of a finite-difference shooting procedure followed by a relaxation process coupled to an adaptive multidomain Chebyshev spectral method [9]. This algorithm introduces two numerical parameters $\xi_f$ and $\xi_s$ ($\xi_s$ represents the shock wave front, $\xi_f$ is defined such that $[\xi_f, \xi_s]$ is the conduction-negligible region), adjusted by the shooting method to obtain the boundary value parameters $B_p$ and $B_\phi$.

**Results**

Similarity solutions depend on five dimensionless flow parameters ($\gamma$, $\mu$, $\nu$, $B_p$, $B_\phi$), which characterize the material, non-linear conduction and external constraints (heat flux and pressure). Given a LMJ-configuration ablation flow [6], we obtain the five dimensionless parameters of a self-similar solution, which is then computed. Fig 1 represents reduced function profiles of density, velocity and temperature. One recognizes (i) the undisturbed fluid region ($\xi > \xi_s$), (ii) the ablation layer (steep density and temperature gradients), (iii) the quasi-isentropic compression region between the ablation layer and the shock-wave front, and (iv) the conduction-dominated region extending from the origin up to the ablation layer. Self-similar solutions for various values of the boundary value parameters $B_p$ and $B_\phi$, and of the fluid adiabatic exponent $\gamma$ have been successfully computed (see Fig. 2 for $\gamma = 5/3$). Variations in laser intensity, fluid density and heat conductivity are accessible with Eq. (6) through proper choices of the boundary value parameters $B_p$ and $B_\phi$. The knowledge of the parameter variation domain allows us to define solutions that could be reached in experiments.

**References**

Figure 2: Reachable values of $B_p$ and $B_\phi$, for $\gamma = 5/3$ and $(\mu, \nu) = (0, 5/2)$. Symbol $\oplus$ indicates an LMJ reference solution (see Fig. 1). Black lines correspond to $\xi_s = \text{cste}$, and blue lines to $\xi_f = \text{cste}$.


