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# An accurate Riemann solver for Godunov methods accounting for relativistic effects

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**Abstract**

We analyze a discontinuity decay in an ideal gas, accountig for relativistic effects in a flat Minkowski space-time. As in the classical case, we solve the Riemann problem through an iterative Newton method, which finds pressure and velocity in the intermediate state. Here, the dicussion is focused on the initialization tecniques and on the primary variables reconstructions for the Godunov scheme.

## 1 Introduction

After the analytical solution of the Riemann problem [1], Riemann solvers became a key tool for shock-capturing methods in numerical relativistic hydrodynamics [2]. Recently, also the Riemann problem in presence of tangential velocity has been solved [3], leading to the possibility to develop multi-dimensional splitting techniqes. In the present paper, the numerical solution of the $1D$ equations will be addressed through a Godunov method.

By using rest mass density $\rho_0$, velocity $u$ (nondimensionalized with the light velocity), Lorentz factor $w = 1/\sqrt{1-u^2}$, pressure $p$, specific entalpy $h$ and total energy $\mu = 1 + h$ combined in the integration variables $D = \rho_0 w$, $S = Dw\mu u$ and $\tau = Dw\mu - p - D$, the $1D$ equations of motion are the following ones [4]:

$$\begin{cases} \partial_t D + \partial_x Du & = & 0 \\ \partial_t S + \partial_x (Su + p) & = & 0 \\ \partial_t \tau + \partial_x (S - Du) & = & 0 \\ h = \gamma p/[(\gamma-1)\rho_0] \, . \end{cases} \tag{1}$$

Hereafter, the integration variables and their fluxes will be considered in the vectors $\boldsymbol{U} = (D, S, \tau)$ and $\boldsymbol{F} = (Du, Su+p, S-Du)$, leading to the compact form $\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F} = \boldsymbol{0}$ of the system (1). The analysis of $1D$-motion of simple waves generated by the decay of a discontinuity (Riemann problem) may be carried out in the $(u,p)$-plane, as it will be shown in the next section.

# 2 Relativistic Riemann problem in the $(u, p)$-plane

The decay of a discontinuity in an ideal fluid leads to the rising of three waves, two of them are rarefactions or shocks and the third one is a contact discontinuity. Four uniform states are identified between the waves, which may be analyzed in the $(u, p)$-plane, as in the classical gasdynamics. Here, any possible relation $f(u, p) = 0$ between velocity $u$ and pressure $p$ *behind* a wave (state 4) is represented (solid lines in Fig. 1), for a specified *ahead* state 1. These curves separate four regions for $(u_4, p_4)$, in which two shock waves ($SCS$), a shock and a rarefaction wave ($SCR$ or $RCS$) and two rarefaction waves ($RCR$) are obtained. In order to find the intermediate states (2 and 3), the curve $g(u, p) = 0$ (dashed line in Fig. 1) which relates $u$ and $p$ in the *ahead* state may be also considered, for the fixed *behind* state $(u_4, p_4)$.

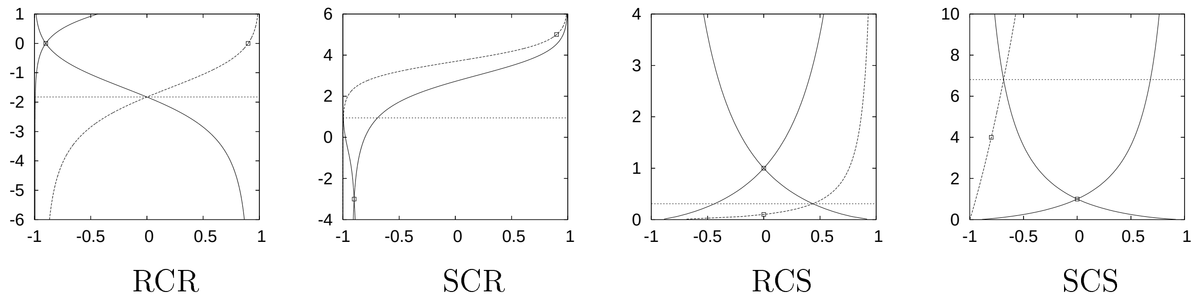


Figure 1: $(u, p)$-plane ($p$ is in logarithmic scale in the first two columns) for the four test cases that will be shown in Fig. 2. A dashed line $p \equiv p_{2,3}$ is also drawn.

The initialization of a Newton method for finding pressure and velocity in the intermediate state may be discussed in the $(u, p)$-plane, too. In the classical case this step is quite obvious: it is generally addressed by using homoentropic conditions. On the contrary, it resuls a non trivial task in the relativistic framework. Two procedures are proposed, using secants or tangents to the above curves.

# 3 First order Godunov scheme

The exact Riemann solver is used to integrate equations (1), through a Godunov scheme. The spatial domain is divided in a certain number of cells, in which space-mean values $\overline{\boldsymbol{U}}$ of the integration variables are only considered. Given $\overline{\boldsymbol{U}}$ at time $t^k$, the $\overline{\boldsymbol{U}}$ at the new time $t^k + \Delta t = t^{k+1}$ is obtained as:

$$\overline{\boldsymbol{U}}(t^{k+1}) = \overline{\boldsymbol{U}}(t^k) - \frac{\Delta t}{\Delta x}\left(\boldsymbol{F}_b - \boldsymbol{F}_a\right),$$

where $\Delta x$ is the length of the cell and $\boldsymbol{F}_{b,a}$ are the fluxes, evaluated at the two ends of the same cell by solving local Riemann problems. Sample cases are shown in Fig. 2.

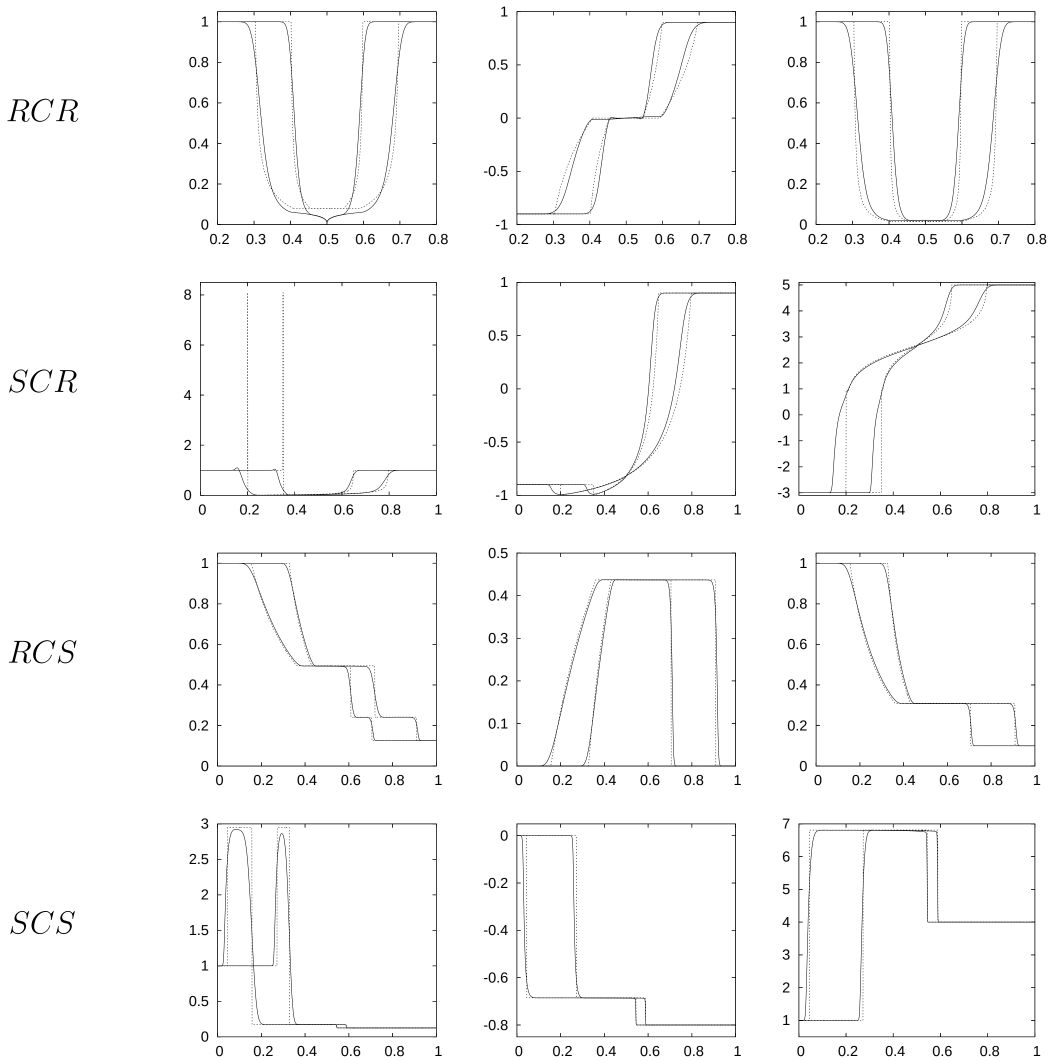


Figure 2: Numerical solutions (solid lines) of the Riemann problem, compared with the corresponding analytical ones (dashed lines), are drawn for waves of kind $RCR$ (times 0.1 and 0.2), $SCR$ (0.15, 0.30), $RCS$ (0.25, 0.50) and $SCS$ (0.25, 0.50). $\rho_0$, $u$ and $p$ are shown vs. $x$ in the first, second and third columns, respectively. The $p$ scale in the SCR case is logarithmic and the states have been specified in Fig. 1.

# 4 Primitive variables reconstruction

A problem raising in Godunov relativistic numerical scheme of Section 3 lies in recovering the primitive variables $u$, $p$ and $\rho_0$ by the integration ones ($D$, $S$ and $\tau$). Two different procedures in velocity and pressure are proposed and tested, both based on the analytical solution of a fourth order algebraic equation. The first one uses the positions $u = \sin\theta$ and $y = \tan(\theta/2)$, which lead to the equation in $y$:

$$S(1-\alpha)y^4 - 2(\tau' + \alpha D)y^3 + 2S(\alpha+1)y^2 - 2(\tau' - \alpha D)y - (1-\alpha) = 0\,, \qquad (2)$$

where $\alpha = (\gamma - 1)/\gamma$ and $\tau' = \tau + D$. Equation (2) has only one root which is physically admissible. Another procedure uses the new variable $\eta = \tau' + p$, for which the following equation holds:

$$(\alpha'-1)^2\eta^4 - 2\alpha'(\alpha'-1)\tau'\eta^3 + [\alpha'^2\tau'^2 + 2(\alpha'-1)S^2 - D^2]\eta^2 - 2\alpha'\tau' S^2\eta + S^2(D^2+S^2) = 0\,, \qquad (3)$$

$\alpha'$ being $1/\alpha$. Equation (3) has two roots physically admissible ($\eta > 0$), but only one of them still gives the correct values of $D$, $S$ and $\tau$.

At the present time, the effects of tangential velocities on the Riemann problem, as well as on the Godunov solver, are under investigation. The next step will be the implementation of a splitting procedure, extending the above numerical technique to $3D$ relativistic flows.

# References


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