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Weakly relativistic dielectric tensor for arbitrary wavenumbers

Francesco Volpe

Max-Planck-Institut für Plasmaphysik, EURATOM-Association, Greifswald, Germany

In approximating Trubnikov's dielectric tensor ε [1], the $\beta_T = \sqrt{2k_BT/mc^2} \ll 1$ limit is often accompanied by the assumptions of quasi-perpendicular incidence [2-11] and small [7-10] or large [11] Larmor radius parameter $\lambda = \frac{1}{2}\left(\frac{N_\perp \beta_T}{Y}\right)^2$, where $N_\perp$ is the refractive index perpendicular to an external magnetic field of normalised magnitude $Y = \omega_c/\omega$. On the other hand, electron Bernstein waves (EBWs) are characterised by $\lambda \gtrsim 1$ and are excited/detected at finite $N_{||}$ [12-15], or tend to develop finite $N_{||}$ anyway [16]. To describe the propagation and absorption of EBWs and of oblique electromagnetic electron cyclotron waves of arbitrary $N_{||}$, a mildly relativistic formulation of ε is derived here with no assumptions on $N_{||}$ and $N_\perp$ except for those which indirectly follow from $\beta_T \ll 1$. To start with, the steady-state solution of the linearized Vlasov-Maxwell problem for a uniformly magnetized plasma is [1, 3]:

$$\varepsilon_{ij} = \delta_{ij} + \frac{2i}{\beta_T^2}\frac{\omega_p^2}{\omega^2}\int_0^\infty d\tau T_{jk}^{(1)} \int \frac{d^3\mathbf{u}}{\gamma} u_i u_k f(\mathbf{u}) \exp(i\gamma\tau - i\mathcal{N}\cdot\mathbf{u}) \tag{1}$$

where the convention of implicitly summing on repeated indices is adopted, ω_p is the plasma frequency, the time τ is renormalized to the wave period ω^{-1}, $\mathbf{u} = \mathbf{p}/mc$ is the normalized momentum, $\gamma = \sqrt{1+\mathbf{u}^2}$ the Lorentz factor and

$$T^{(1)} = \begin{pmatrix} \cos Y\tau & -\sin Y\tau & 0 \\ \sin Y\tau & \cos Y\tau & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

The axes are chosen to yield $\mathbf{N} = (N_\perp, 0, N_{||})$. Finally,

$$\mathcal{N}_x = \frac{1}{Y} N_\perp \sin Y\tau, \qquad \mathcal{N}_y = \frac{1}{Y} N_\perp (\cos Y\tau - 1), \qquad \mathcal{N}_z = N_{||}\tau \tag{3}$$

and, for a thermal relativistic plasma, the distribution function is the relativistic maxwellian

$$f(\mathbf{u}) = \frac{\exp(-2\gamma/\beta_T^2)}{2\pi\beta_T^2 K_2(2/\beta_T^2)} \tag{4}$$

where K_2 is the modified Bessel function of the second kind of order 2, also known as MacDonald function [18]. Integration over momenta gives [1]:

$$\varepsilon_{ij} = \delta_{ij} + 4i\frac{\omega_p^2}{\omega\omega_c}\frac{1}{\beta_T^4 K_2}\int_0^\infty d\tau \left[T_{ij}^{(1)} \frac{K_2(R^{1/2})}{R} - T_{ij}^{(2)} \frac{K_3(R^{1/2})}{R^{3/2}} \right] \tag{5}$$

where $R = 4b^2 + \mathcal{N}^2$, $b = \frac{1}{\beta_T^2} - i\frac{\tau}{2}$ and $T_{ij}^{(2)} = \mathcal{N}_i T_{jk}^{(1)} \mathcal{N}_k$.

The limit $\beta_T \to 0$ implies $b \to \infty$, that justifies the asymptotic expansions [18]

$$\frac{K_2(R^{1/2})}{R} \simeq e^{-R^{1/2}}\sqrt{\frac{\pi}{2}}\left[\frac{1}{R^{5/4}} + \frac{15}{8}\frac{1}{R^{7/4}}\right] \tag{6}$$

$$\frac{K_3(R^{1/2})}{R^{3/2}} \simeq e^{-R^{1/2}}\sqrt{\frac{\pi}{2}}\left[\frac{1}{R^{7/4}} + \frac{35}{8}\frac{1}{R^{9/4}}\right] \tag{7}$$

and their further approximation to the highest orders in b. This leads to:

$$\varepsilon_{ij} = \delta_{ij} + i\frac{\omega_p^2}{\omega^2}\frac{1}{\beta_T^4 K_2(2\beta_T^{-2})}\frac{\pi^{1/2}}{2}\int_0^\infty d\tau \frac{e^{-2b}e^{-\mathcal{N}^2/4b}}{16b^{11/2}}Q_{ij} \tag{8}$$

where

$$Q_{ij} = T_{ij}^{(1)} b(16b^2 + 15b + 4\mathcal{N}^2) - T_{ij}^{(2)}(8b^2 + 17.5b + \mathcal{N}^2) \tag{9}$$

Note that approximating the exact integral, eq.5, is more rigorous than Taylor-expanding γ up to the second order in u in the integrand in eq.1, which gives the same as eqs.8-9 but with slightly different coefficients for eq.9 [20].

After the changes of variables $\beta_T^2\tau/2 \to t$ and $2Y/\beta_T^2 \to y$ and after expanding:

$$\exp\left[\frac{\lambda\cos yt}{1-it}\right] = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\left[\frac{\lambda}{2(1-it)}\right]^{p+q}\frac{e^{i(p-q)yt}}{p!q!} = \sum_{n=-\infty}^{\infty}\sum_{m=|n|}^{\infty}\frac{(\lambda/2)^m}{\left(\frac{m+n}{2}\right)!\left(\frac{m-n}{2}\right)!}\frac{e^{inyt}}{(1-it)^m}, \tag{10}$$

with the double sum restricted to even values of $m+n$, all integrals in eq.8 take the form

$$\mathcal{Q}_{mn,ij} = e^{-\lambda}\beta_T^6\int_0^\infty dt \frac{Q_{ij}}{(1-it)^{m+11/2}}\exp\left[\frac{2}{\beta_T^2}(nY+1)it - \frac{\lambda + N_{||}^2 t^2/\beta_T^2}{1-it}\right] \tag{11}$$

involving generalized Shkarofsky functions [9, 21, 22]

$$\mathcal{F}_{q,r}(z,a) = -i\int_0^\infty \frac{(it)^r}{(1-it)^q}\exp\left[izt - \frac{at^2}{1-it}\right]dt, \tag{12}$$

of shifted arguments, that will be omitted for brevity and summarized by an index for the harmonic number:

$$\mathcal{F}_{q,r,n} = \mathcal{F}_{q,r}\left(\frac{2}{\beta_T^2}(nY+1) - \lambda,\ \frac{N_{||}^2}{\beta_T^2} - \lambda\right). \tag{13}$$

Substituting in eq.11 an approximation up to $\mathcal{O}(\beta_T^4)$ of eq.9 and replacing, for consistency, K_2 at denominator of eq.1 with its asymptotic limit, yield the following expression for the slightly relativistic dielectric tensor:

$$\varepsilon_{ij} = \delta_{ij} + i\frac{\omega_p^2}{\omega^2}\frac{e^\lambda}{16+15\beta_T^2}\sum_{n=-\infty}^{\infty}\sum_{m=|n|}^{\infty}\frac{(\lambda/2)^m \mathcal{Q}_{mn,ij}}{\left(\frac{m+n}{2}\right)!\left(\frac{m-n}{2}\right)!} \tag{14}$$

where, as usual, the summation is restricted to even values of $m+n$, and

$$\mathscr{Q}_{mn,11} = 8i(1-\underline{N_{||}^2 O_{2,2,0}})\mathscr{F}^+_{m+\frac{5}{2},0,n\pm1} + \underline{\frac{15}{2} i\beta_T^2 \mathscr{F}^+_{m+\frac{7}{2},0,n\pm1}} \quad (15)$$

$$+ \; i\beta_T^2 \frac{N_\perp^2}{Y^2}(2-\underline{N_{||}^2 O_{2,2,0}})\left(\mathscr{F}^+_{m+\frac{7}{2},0,n\pm2} - 2\mathscr{F}_{m+\frac{7}{2},0,n}\right) \quad (16)$$

$$\mathscr{Q}_{mn,22} = 8i(1-\underline{N_{||}^2 O_{2,2,0}})\mathscr{F}^+_{m+\frac{5}{2},0,n\pm1} + \underline{\frac{15}{2} i\beta_T^2 \mathscr{F}^+_{m+\frac{7}{2},0,n\pm1}} \quad (17)$$

$$- \; i\beta_T^2 \frac{N_\perp^2}{Y^2}(2-\underline{N_{||}^2 O_{2,2,0}})\left(\mathscr{F}^+_{m+\frac{7}{2},0,n\pm2} - 4\mathscr{F}^+_{m+\frac{7}{2},0,n\pm1} + 6\mathscr{F}_{m+\frac{7}{2},0,n}\right) \quad (18)$$

$$\mathscr{Q}_{mn,21} = 8(1-\underline{N_{||}^2 O_{2,2,0}})\mathscr{F}^-_{m+\frac{5}{2},0,n\pm1} + \underline{\frac{15}{2} \beta_T^2 \mathscr{F}^-_{m+\frac{7}{2},0,n\pm1}} \quad (19)$$

$$+ \; \beta_T^2 \frac{N_\perp^2}{Y^2}(2-\underline{N_{||}^2 O_{2,2,0}})\left(\mathscr{F}^-_{m+\frac{7}{2},0,n\pm2} - 2\mathscr{F}^-_{m+\frac{7}{2},0,n\pm1}\right) \quad (20)$$

$$\mathscr{Q}_{mn,33} = \frac{16i}{\beta_T^2} N_{||}^2(2-\underline{N_{||}^2 O_{2,2,0}})\mathscr{F}_{m+\frac{7}{2},2,n} + 2i(8+\underline{27N_{||}^2 O_{2,2,0}})\mathscr{F}_{m+\frac{5}{2},0,n} \quad (21)$$

$$+ \; \underline{15i\beta_T^2 \mathscr{F}_{m+\frac{7}{2},0,n}} - \underline{4i\beta_T^2 \frac{N_\perp^2}{Y^2} N_{||}^2 \left(\mathscr{F}^+_{m+\frac{11}{2},2,n\pm1} - 2\mathscr{F}_{m+\frac{11}{2},2,n}\right)} \quad (22)$$

$$\mathscr{Q}_{mn,13} = 4i(2-\underline{N_{||}^2 O_{2,2,0}})\frac{N_\perp}{Y} N_{||} \mathscr{F}^-_{m+\frac{7}{2},1,n\pm1} + \underline{\frac{35}{2} i\beta_T^2 \frac{N_\perp}{Y} N_{||} \mathscr{F}^-_{m+\frac{9}{2},1,n\pm1}} \quad (23)$$

$$\mathscr{Q}_{mn,23} = 8(2-\underline{N_{||}^2 O_{2,2,0}})\frac{N_\perp}{Y} N_{||} \mathscr{F}_{m+\frac{7}{2},1,n} + \underline{35\beta_T^2 \frac{N_\perp}{Y} N_{||} - 2\mathscr{F}_{m+\frac{9}{2},1,n}} + i\mathscr{Q}_{mn,13} \quad (24)$$

$\mathscr{Q}_{mn,12} = -\mathscr{Q}_{mn,21}$, $\mathscr{Q}_{mn,31} = \mathscr{Q}_{mn,13}$ and $\mathscr{Q}_{mn,32} = -\mathscr{Q}_{mn,23}$. The compact notations

$$\mathscr{F}^+_{q,r,n\pm p} = \mathscr{F}_{q,r,n+p} + \mathscr{F}_{q,r,n-p} \quad (25)$$

$$\mathscr{F}^-_{q,r,n\pm p} = \mathscr{F}_{q,r,n+p} - \mathscr{F}_{q,r,n-p} \quad (26)$$

were employed for some frequently occurring sums and differences and an operator on the subscripts q and r, defined by $O_{2,2,0}\mathscr{F}_{q,r,n} = \mathscr{F}_{q+2,r+2,n}$, was introduced.

In eq.13 dispersion was found to depend on Larmor radius, in agreement with relativistic eq.1. The shift of generalized Shkarofsky functions' arguments z and a by an amount λ can be interpreted as a linear finite-Larmor-radius correction to the resonance condition through $z = 2(nY+1)/\beta_T^2$ and to its width in inhomogenoeus plasmas, through the ratio of Doppler to relativistic width, $a = N_{||}/\beta_T$. The physical meaning of $z \to z-\lambda$ is the well-known relativistic downshift.

At this point it should be remembered that it is customary to treat as a constant the Lorentz factor γ in the denominator in the integrand of eq.1 and to Taylor-expand only the factor γ in the exponent, on the ground that relativistic corrections in the denominator have a comparatively small effect on the integral. This is equivalent to approximating $Q_{ij} \simeq (T_{ij}^{(1)} 2b - T_{ij}^{(2)}) 8b^2$

i.e. to suppressing all underlined terms in eqs.15-24, which is only partly legitimate under the assumption $N_{||} \ll \beta_T$ that, by the way, was not adopted in the present work.

The double sum in eq.14 might look computationally expensive, but functions $\mathscr{F}$ of different q (thus, $\mathscr{Q}$ of different m) can recursively be related to one another [2, 5, 8, 10, 21]. Note also that simple Shkarofsky functions of $r = 0$ are the most widely used in eqs.15-24. Generalized functions $\mathscr{F}_{q,r}$ with $r \neq 0$ can be expressed in terms of $\mathscr{F}_q$ [17] and this be related to the classical dispersion function Z [8]. The latter coincides with the Dawson's integral, for the evaluation of which plenty of routines are available. Besides, for relativistically broadened but well-resolved lines, the double sum can be restricted to diagonal terms ($m = n$).

An alternative form of eq.14 can be obtained by recognizing, in the sum over m in eq.10, the generating function for the modified Bessel function of the first kind, I_n. This generalizes a standard Bessel function identity utilized in the derivation of the warm non-relativistic dielectric tensor [19]. However, it does so by replacing $\lambda \rightarrow \lambda/(1-it)$, i.e. by introducing a time dependence for I_n that cannot be factored out of the integral anymore. The resulting expression,

$$\varepsilon_{ij} = \delta_{ij} + 2i\frac{\omega_p^2}{\omega^2}\frac{\beta_T^4}{16+15\beta_T^2}\sum_{n=-\infty}^{\infty}\int_0^\infty dt\,\frac{Q_{ij}}{(1-it)^{11/2}}\exp\left[\frac{2}{\beta_T^2}(nY+1) - \frac{\lambda + N_{||}^2 t^2/\beta_T^2}{1-it}\right] I_n\left(\frac{\lambda}{1-it}\right) \quad (27)$$

involves integrals more complicated than eq.11. On the other hand, it avoids the nuisance of the double sum and can be useful when a high m is required for convergence.

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