THE SPECTRUM OF TWO-DIMENSIONAL INCOMPRESSIBLE MAGNETOHYDRODYNAMIC FLOWS

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1. Introduction

Waves and instabilities in plasmas may be investigated by means of spectral analysis. Up to now our investigations of the spectrum of plasmas with equilibrium flow mainly have been restricted to one-dimensional equilibria [1, 2]. The few results on the spectrum of two-dimensional flow equilibria, however, show that new phenomena may occur. For instance, equilibrium flow in two-dimensional geometries may turn the continuous spectrum overstable [3]. Therefore, it is interesting to investigate the combined effects of magnetic field, flow, and geometry on the spectrum [4].

Before one is able to calculate the spectrum of a plasma equilibrium state, one first has to find this state. Since the equilibrium is two-dimensional it has to be found numerically. In order to do this the FLow Equilibrium Solver (FLES) program is developed with which we are able to find equilibrium states in a cylinder with non-circular cross section. These cylinder geometries are a model for flux tubes like tokamaks, coronal loops, astrophysical jets, etc.

When the equilibrium state is found the spectrum can be calculated. For this we developed the INcompressible 2 Dimensional FLow Eigenvalue Solver (IN2FLES) program. With this program the effect of the geometry on the spectrum is investigated. The first results of this investigation are presented here.

2. Equilibrium

Consider a two-dimensional, stationary, incompressible MHD equalibrium in a cylindrical geometry with non-circular cross-section. Such an equilibrium is translationally symmetric along the z-axis. The equilibrium flow, $\mathbf{V}_0$, and the equilibrium magnetic field, $\mathbf{B}_0$, may then be written as:

$$\mathbf{V}_0 = \frac{1}{\rho}\mathbf{e}_z \times \nabla\chi + V_z\mathbf{e}_z, \qquad \mathbf{B}_0 = \mathbf{e}_z \times \nabla\psi + B_z\mathbf{e}_z. \tag{1}$$

The density, ρ, the longitudinal components of the velocity and the magnetic field, resp. V_z and B_z, and the potential for the poloidal flow, χ, are functions of the magnetic flux, ψ, only. Then the MHD equations reduce to the transfield or modified Grad-Shafranov (MGS) equation [3]:

$$(1 - M^2)\nabla \cdot \nabla\psi - \tfrac{1}{2}(M^2)'|\nabla\psi|^2 + H' = 0, \tag{2}$$

where $' \equiv d/d\psi$ and

$$M^2(\psi) \equiv \chi'^2/\rho, \qquad H(\psi) \equiv P + \tfrac{1}{2}\rho V_p^2 + \tfrac{1}{2}B_z^2. \tag{3}$$

From this equation the magnetic flux can be solved whenever the profiles for the poloidal Alfvén Mach number (M) and the Bernoulli function (H) are prescribed.

For elliptical cross-sections and constant profiles for M and H' the MGS equation has an analytical solution:

$$\psi = x^2 + \left(\frac{y}{c}\right)^2; \qquad c \equiv b/a. \tag{4}$$

For more general cross-sections or more general profiles the MGS equation has to be solved numerically. For this the FLES program is developed. The domain is discretised using a finite element method exploiting bicubic Hermite elements. The solution for the magnetic flux is then found by a Picard iteration [4]. Finally, the solution is put onto straight field line coordinates (ψ, ϑ, z), which allows for an easy representation of the $\mathbf{B}_0 \cdot \nabla$ and $\mathbf{V}_0 \cdot \nabla$ operators, which arise in the calculation of the spectrum of these equilibria.

3. Spectrum

When an equilibrium solution is found the corresponding spectrum may be found by linearising the MHD equations about the equilibrium:

$$f(\mathbf{x}, t) = f_0(\sqrt{\psi}, \vartheta) + e^{\sigma t} \sum_{m=M_1}^{M_2} f_{1,m}(\sqrt{\psi}) e^{im\vartheta + ik_z z}. \tag{5}$$

The linearised MHD equations are then approximated by:

$$\sigma\rho_1 = -\mathbf{V}_0 \cdot \nabla\rho_1 - \mathbf{v}_1 \cdot \nabla\rho_0, \tag{6}$$

$$\begin{aligned} \sigma\rho_0\mathbf{v}_1 &= -\rho_0\mathbf{V}_0 \cdot \nabla\mathbf{v}_1 - \rho_0\mathbf{v}_1 \cdot \nabla\mathbf{V}_0 - \rho_1\mathbf{V}_0 \cdot \nabla\mathbf{V}_0 \\ &\quad + (\nabla \times \mathbf{A}_1) \cdot \nabla\mathbf{B}_0 + \mathbf{B}_0 \cdot \nabla(\nabla \times \mathbf{A}_1) - \nabla\pi_1, \end{aligned} \tag{7}$$

$$\sigma\mathbf{A}_1 = \mathbf{V}_0 \times (\nabla \times \mathbf{A}_1) + \mathbf{v}_1 \times \mathbf{B}_0, \tag{8}$$

$$\sigma\pi_1 = Z\nabla \cdot \mathbf{v}_1. \tag{9}$$

Here all subscripts 0 indicate equilibrium quantities, all subscripts 1 indicate perturbed quantities, and π_1 is the perturbed total pressure. The perturbed magnetic field is replaced by a vector potential, $\mathbf{b}_1 = \nabla \times \mathbf{A}_1$, making it divergence free. The incompressibility condition of the perturbations is replaced by equation (9). This allows us to treat this condition the same way as the other equations. Note that a new set of eigenvalues is introduced which are of the order Z. The other eigenvalues are approximated up to order $1/Z$. Hence, by taking Z to be a large

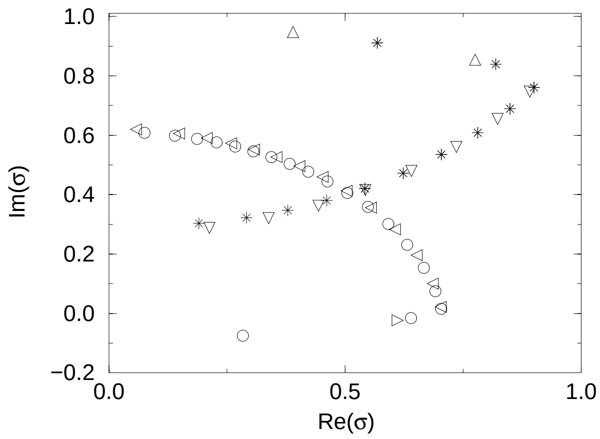

Fig. 1: The effect of ellipticity on the overstable spectrum for a rotating plasma, $M = 0.6$. Shown is the overstable spectrum for even modes and odd modes in a circular and an elliptical cylinder. Circular cross-section: even modes $m = -2$ ($\triangle$), $m = 0$ ($\bigtriangledown$); odd modes $m = -1$ ($\triangleleft$), $m = 1$ ($\triangleright$). Elliptical cross-section: even modes ($*$); odd modes ($\circ$).

number, typically $Z \sim 10^6$, the extra eigenvalues are easily filtered out and the true eigenvalues are well approximated.

The linearised MHD equations are discretised using a mixed finite element / Fourier method. This gives rise to a very large eigenvalue problem that may be solved using a Jacobi-Davidson solver. All this is implemented in the IN2FLES program.

The continuous spectrum may be treated seperately. For equilibrium (4) it decouples into a longitudinal and a tangential part, which coincide for a cylinder with a circular cross-section. For non-circular cross-sections the longitudinal part remains unchanged, whereas the tangential part is modified. This modification lifts the degeneracy and, moreover, yields that this part of the continuous spectrum may turn overstable for certain values of M.

The discrete spectrum is calculated using the IN2FLES program. Changing the equilibrium (4) from a circular cross-section, $c = 1$, to an elliptical cross-section, $c = 1.5$, modifies the overstable discrete spectrum. Individual modes get a larger growth rate. However, the growth rate of the most unstable mode is hardly affected. This is shown on Fig. 1.

For large enough Mach numbers a new instability arises due to the non-circularity of the geometry and the background flow. This instability is explained by a coupling between the Alfvén continuum and the periodic rotation and this may eventually lead to an unstable continuous spectrum or cluster point, see Fig. 2.

4. Conclusions

- For linear equilibria the continuous spectrum decouples into a longitudinal part and a tangential part. The longitudinal part is independent of the cross-section. The tangential part may turn overstable.

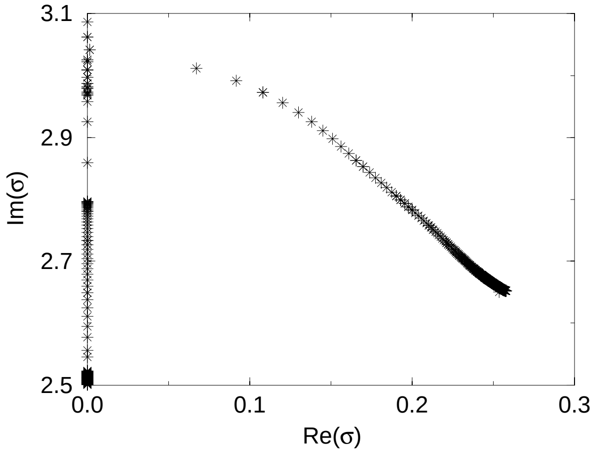

Fig. 2: Overstable cluster spectrum for an elliptical cross-section: $c = 1.5$, $M = 1.5$, $k_z = 10$, $B_z = 0.08$, and $V_z = 0$.

- The discrete spectrum for linear equilibria is modified when the cross-section changes from circular to elliptical, but the growth rate of the most unstable mode is hardly affected (Fig. 1).
- A non-circular cross-section in combination with a background flow may lead to an overstable cluster point (Fig. 2).

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References

[1] R.J. Nijboer, A. Lifschitz, and J.P. Goedbloed: "Spectrum and stability of a rigidly rotating compressible plasma", J. Plasma Physics **58** (1997) 101-121.

[2] R.J. Nijboer, B. v.d. Holst, S. Poedts, and J.P. Goedbloed: "Calculating magnetohydrodynamic flow spectra", Comp. Phys. Commun. **106** (1997), 39-52.

[3] R.J. Nijboer, J.P. Goedbloed, and A. Lifschitz: "The spectrum of MHD flows about X-points", *accepted for publication in* J. Plasma Physics.

[4] R.J. Nijboer, and J.P. Goedbloed: "Mode coupling in two-dimensional magnetohydrodynamic flows", *submitted to* J. Plasma Physics.