CALCULATIONS OF FEEDBACK STABILIZATION OF WALL MODES

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I. Introduction

Important ideal MHD modes grow slowly when a conducting wall surrounds a plasma. Feedback stabilization of these modes may be required for tokamaks and other plasma confinement concepts to obtain adequate plasma pressure and self-driven current for practical fusion power.

The basic concepts and equations that are required to simulate the feedback stabilization of ideal plasmas will be given. The information that is required about the plasma is the eigenvalues of δW, the energy change produced by a perturbation, and the angular dependence of the eigenmodes on the plasma surface. In addition to the fundamental equations, simple examples that explore feedback requirements will be developed. This work has been accepted for publication in the Physics of Plasmas.

II. Specification of Perturbation

When calculating the magnetic perturbation outside of the plasma, the required boundary condition on the plasma surface is the normal field

$$\mathcal{J}\mathbf{b}\cdot\nabla r = \sum \Phi_i f_i(\theta,\varphi)$$

with $\mathcal{J}$ the Jacobian of (r, θ,φ) coordinates. These coordinates are to a large extent arbitrary, but the plasma boundary is assumed to be a constant r surface. The functions $f_i(\theta,\varphi)$ are eigenfunctions of δW, which are normalized by $\int f_i f_j d\theta d\varphi = \delta_{ij}$. The energy that would be required to produce a perturbation in the absence of currents outside the plasma is

$$\delta W = \sum w_i \Phi_i^2/2.$$

The perturbation amplitudes, Φ_i, satisfy $\Phi_i = \int f_i(\theta,\varphi)\mathbf{b}\cdot d\mathbf{a}$ and have units of flux.

In the region outside of the plasma, currents in plasma that are associated with the perturbation can be represented by a surface current

$$\mathbf{K} = -\nabla r X \nabla\kappa(\theta,\varphi).$$

The current potential $\kappa(\theta,\varphi)$ can be written in terms of eigenfunctions $g_i(\theta,\varphi)$ and amplitudes I_i which have units of Amperes,

$$\kappa = \sum I_i g_i(\theta,\varphi).$$

The g_i are normalized so $\int g_i f_i d\theta d\varphi = 1$. Each g_i would produce the field $\mathcal{J}b\cdot\nabla r = \Phi_i f_i(\theta,\varphi)$ if there were no other currents. The inductance of a mode is defined by

$L_i = \Phi_i/I_i$.

The plasma current associated with the perturbation I_i is proportional to the normal magnetic field of the perturbation at the unperturbed the plasma surface. This proportionality can be written

$$L_i I_i = \sum(\delta_{ii'} + s_i \lambda_{ii'}) \Phi_{i'}.$$

One can show that the stability constant s_i is given by the associated eigenvalue of δW. The precise relation is $s_i = -w_i L_i$. A perturbation is unstable without a wall when s_i is positive. The matrix $\lambda_{ii'}$ is given by $\lambda^{-1}{}_{ii'} = \int f_i g_{i'}\, d\theta d\varphi$.

External currents, in the wall and in feedback circuits, produce flux on the plasma surface. These contributions can be written as

$$\int f_i(\theta,\varphi)\, \mathbf{b_x}\cdot d\mathbf{a} = \sum M_{iw} I_w$$

with I, the current in one of these circuits and M_{iw} a mutual inductance. The perturbation flux at plasma surface due to both currents within the plasma, I_i, and due to external circuits is

$$\Phi_i = L_i I_i + \sum M_{iw} I_w.$$

The wall and coil currents are determined by an induction equation

$$d\Phi_w/dt = - R_w I_w + V_w(t)$$

with the flux through wall or coil circuits

$$\Phi_w = \sum L_{ww'} I_{w'} + \sum M_{wi} I_i.$$

The voltage that is supplied to the feedback coils is $V_w(t)$.

The equations that have been given can be used to develop a general computational method.

III. Single Mode Example

The properties of these equations can be illustrated by assuming that there is a single plasma mode and by simplifying the wall and external circuits to the maximum extent possible. This model has four perturbed fluxes: the plasma Φ, the conducting wall Φ_w, the feedback Φ_f, and the sensor Φ_s. The voltage on the feedback circuit is determined by the flux through the sensor loop. By suitably defining the dimensionless coefficients k_p and k_d, the feedback voltage can be given in the form

$$V_f(t) = -\frac{L_f}{M_{pf}}\left(\gamma_w k_p + k_d \frac{d}{dt}\right)\Phi_s.$$

The strength of proportional feedback is measured by k_p and the strength of derivative feedback by k_d. The current in the feedback circuit obeys

$$\tau\frac{d^2I_f}{dt^2} + \frac{dI_f}{dt} + \gamma_f I_f = -\left(\gamma_w k_p + k_d\frac{d}{dt}\right)\frac{\Phi_s}{M_{pf}}.$$

The time delay of the circuit is τ, which is assumed small, and $\gamma_f \equiv R_f/L_f$. The single mode plasma equations are $I = (1 + s)\,\Phi/L$ and $\Phi = L\,I + M_{pw}\,I_w + M_{pf}\,I_f$, which give the perturbed plasma current, I, as a function of the wall and feedback currents, I_w and I_f. The current in the conducting wall obeys $\Phi_w = L_w I_w + M_{wp}I_p + M_{wf}I_f$ with $d\Phi_w/dt = -\gamma_w L_w I_w$. These equations can be written as

$$\mathcal{L}_{ww}\frac{dI_w}{dt} + \mathcal{L}_{wf}\frac{dI_f}{dt} = -\gamma_w L_w I_w.$$

The effective self-inductance is

$$\mathcal{L}_{ww} = -L_w D(s),$$

and D(s) is the discriminant for stability,

$$D(s) \equiv \frac{1+s}{s}c - 1.$$

The coupling between the wall and the plasma is $c \equiv M_{wp}M_{pw}/LL_w$. The stability discriminant must be positive for stabilization by an ideal wall; if D is negative with s positive the perturbation grows on an Alfvénic time scale. If s is negative there is stability. If both s and D are positive but there is no feedback, $I_f = 0$, a mode grows at the rate γ_w/D. The effective mutual inductance between the feedback and the wall is

$$\mathcal{L}_{wf} = -L_w\frac{M_{pf}}{M_{pw}}\left(D+c_f\right)$$

and the relative coupling of the feedback to the plasma and to the wall is

$$c_f \equiv 1 - \frac{M_{pw}M_{wf}}{L_w M_{pf}}.$$

This coupling must satisfy $c_f \geq 0$ to feedback stabilize values of s up to the ideal wall stability limit, which is $D = 0$ with $s > 0$.

Two strategies of feedback stabilization will be discussed: (1) the sensor measuring the perturbation on the plasma and (2) the sensor measuring the flux that penetrates the wall. For the first strategy one can assume that the magnetic flux through the sensor is multiplied by a suitable constant so the sensor flux equals the perturbed flux on the plasma surface $\Phi_s = \Phi$. Letting I_w and I_f be proportional to $\exp(\nu t)$, one obtains a characteristic polynomial

$$a_3\nu^3 + a_2\nu^2 + a_1\nu + a_o = 0$$

which has coefficients

$$a_0/\gamma_w = -\gamma_f + \gamma_w k_p/s,$$

$$a_1 = D\gamma_f + \gamma_w (k_d/s + c_f k_p/s - 1),$$

$$a_2 = D + c_f k_d/s, \text{ and } a_3 = D\tau.$$

Since the time delay τ is assumed infinitesimal, the cubic equation can be solved. One finds all of the coefficients must have same sign for stability. Since D and τ are positive in the region that can be stabilized by the wall, all four coefficients must be positive. Assume $c_f > 0$, then all four coefficients can be made positive using proportional feedback with $k_p > 0$ sufficiently large.

The second strategy for feedback places the sensor so it measures the flux through the conducting wall. We assume the output of the sensor flux is multiplied by the required constant to make $\Phi_s = (M_{pw}/L_w)\Phi_w$. The characteristic polynomial now has coefficients

$$a_0/\gamma_w = -\gamma_f + \gamma_w (D + c_f) k_p,$$

$$a_1 = D\gamma_f + \gamma_w \{(D + c_f)k_d - 1\},$$

$$a_2 = D, \text{ and } a_3 = D\tau.$$

Again, all the coefficients must be positive for stability, and proportional feedback is required, $k_p > 0$. If $\gamma_f < \gamma_w/D$, derivative feedback is also required, $k_d > 0$.

IV. Discussion

General equations for studying feedback stabilization of ideal wall modes have been developed. Jim Bialek is developing an electromagnetic code for the wall, feedback coils, and sensors, so a full simulation of feedback can be performed. The general equations are formally similar to those for feedback stabilization of vertical instabilities. In addition to the work reported here, a method has been developed for handling resistive plasmas and plasma torques, which will soon be submitted for publication.

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