SYMMETRIES FOR THE ELECTRON MAGNETOHYDRODYNAMICS

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1. Constants of the motion restrict dynamics of the physical system, and their importance for both finding the motion of the system and analysing its stability by use of the variational approach is well known. Construction of the conservation laws for a given set of differential equations is a nontrivial task, and generally there is no regular procedure of finding all possible constants of the motion. For Lagrangean systems there exists the Noether's theorem which provides an algorithm to construct a conservation law for every variational symmetry of the system (i.e., the infinitesimal transformation which does not change the action integral of Lagrangean system). Moreover, it was shown that the complete set of variational symmetries is sufficient to generate *all* conservation laws of Lagrangean system [1]. It is significant that each variational symmetry of the Lagrangean system is a Lie-Bäcklund symmetry of the Euler-Lagrange equations (i.e., the equations arising from a variational problem for the action) and can be found by examining the invariance properties of Euler-Lagrange equations. In this paper we apply the above approach to the equations of electron magnetohydrodynamics (EMHD).

2. EMHD provides a fluid description of the plasma behavior on scales below the ion inertial length c/ω_{pi}, where the plasma dynamics is governed by the electron flow and its selfconsistent magnetic field, while the ions form a static charge neutralizing background [2, 3]. It is formed by the following set of equations:

$$\partial_t \mathbf{\Omega} = \mathbf{\nabla} \times [\mathbf{v} \times \mathbf{\Omega}] + \mathbf{\nabla} T \times \mathbf{\nabla} \eta, \tag{1}$$

$$\mathbf{\nabla} \times \mathbf{B} = a\rho \mathbf{v}, \tag{2}$$

$$\mathbf{\Omega} = \mathbf{\nabla} \times \mathbf{v} + a\mathbf{B}, \tag{3}$$

$$\partial_t \eta + \mathbf{v}\mathbf{\nabla}\eta = 0. \tag{4}$$

Here the first equation is obtained by applying $curl$ to the equation of the motion of the electron fluid, Eq.(2) is Maxwell equation, Eq.(4) is the adiabatic equation. On the whole the symbols are standard, $a = e/m$ is the electron charge-to-mass ratio, η is the specific entropy and T is the temperature of electrons. Plasma is considered to be quasineutral, and since the ions are considered to be fixed, then the electron density does not change in time, $\partial_t \rho = 0$. It is equal to the ion background density, so that $\rho = \rho(\mathbf{x})$. The set of Eqs.(1)–(4) is closed if the temperature is defined as the function of density and entropy, $T = T(\rho, \eta)$, or, in other words, the specific internal energy is defined as the function of entropy and density, $\epsilon = \epsilon(\rho, \eta)$. The temperature is related to ϵ by the equation $T = (\partial\epsilon/\partial\eta)_\rho$.

3. To write the Lagrangian of this set of equations is rather nontrivial task. The main difficulty consists in an adequate choice of the set of variables, which could be considered to be independent. The physical variables, which are involved into EMHD equations, are not independent just due to the equations relating them to each other. Like the cases of hydrodynamics and ideal

MHD [4, 5, 6] the above difficulty is solved by a transition to the Lagrangean description of the electron fluid. Namely, introduce, like [6], three independent $(J = \boldsymbol{\nabla}\alpha^1\ [\boldsymbol{\nabla}\alpha^2 \times \boldsymbol{\nabla}\alpha^3] \neq 0)$ Lagrangean quantities $\alpha^i(t, \mathbf{x})$ $(i = 1 \div 3)$ frozen into the electron fluid:

$$\partial_t \alpha^i + \mathbf{v}\boldsymbol{\nabla}\alpha^i = 0, \quad i = 1 \div 3. \tag{5}$$

It is easy to check that the Jacobian J satisfies the continuity equation, $\partial_t J + \boldsymbol{\nabla} \cdot (J\mathbf{v}) = 0$, and if $J \neq 0$ at $t = 0$ then it is not equal to zero at any t for any smooth velocity field $\mathbf{v}(t, \mathbf{x})$. The contra- and covariant basis vectors corresponding to the Lagrangean coordinates $\{\alpha^i(t, \mathbf{x})\}$ are defined by:

$$\mathbf{e}^i = \boldsymbol{\nabla}\alpha^i, \quad \mathbf{e}_i = \frac{1}{2J}\epsilon_{imn}\mathbf{e}^m \times \mathbf{e}^n \ , \tag{6}$$

where ϵ_{imn} is the completely antisymmetric unit tensor. Hereafter summation over the repeated indices is assumed. All the physical variables describing the electron fluid can be expressed in terms of α^i-s and their first derivatives (see, e.g., [6]):

$$\mathbf{v} = -\mathbf{e}_i \partial_t \alpha^i, \quad \rho = JR(\alpha^i), \quad \eta = \eta(\alpha^i)\ , \tag{7}$$

where the functions R and η are determined by the initial conditions. Constructing the Lagrangian we will assume that in addition to α^i-s the vector potential $\mathbf{A}$, defined by $\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$, is an independent variable too.

Consider an arbitrary transformation of independent variables $\alpha^{*i} = \alpha^i + \delta\alpha^i \equiv \alpha^i - \xi^i$, $\mathbf{A}^* = \mathbf{A} + \delta\mathbf{A}$. The variations of physical variables are expressed in terms of the displacement vector $\boldsymbol{\xi} \equiv \xi^i \mathbf{e}_i$ and $\delta\mathbf{A}$ by the following equations:

$$\begin{aligned} \delta\rho &= -\boldsymbol{\nabla} \cdot (\rho\boldsymbol{\xi}), \quad \delta\mathbf{v} = \partial_t\boldsymbol{\xi} + (\mathbf{v}\boldsymbol{\nabla})\boldsymbol{\xi} - (\boldsymbol{\xi}\boldsymbol{\nabla})\mathbf{v}, \\ \delta\eta &= -\boldsymbol{\xi}\boldsymbol{\nabla}\eta, \quad \delta\mathbf{B} = \boldsymbol{\nabla} \times \delta\mathbf{A}. \end{aligned} \tag{8}$$

Due to the quasineutrality condition the density is defined by the ions, and its variation should be equal to zero, $\delta\rho = 0$. Therefore, the displacement vector $\boldsymbol{\xi}$ should satisfy the equation $\boldsymbol{\nabla} \cdot (\rho\boldsymbol{\xi}) = 0$. Hence instead of $\boldsymbol{\xi}$ an arbitrary vector $\boldsymbol{\chi}$ defined by the equation

$$\boldsymbol{\xi} = \frac{1}{\rho}\boldsymbol{\nabla} \times \boldsymbol{\chi}, \tag{9}$$

should be considered as the independent variation.

Consider the Lagrangian density in the following form:

$$L = \frac{\rho\mathbf{v}^2}{2} - \rho\varepsilon(\rho, \eta) + a\rho\mathbf{v}\mathbf{A} - \frac{\mathbf{B}^2}{2}. \tag{10}$$

It is easy to prove that EMHD equations (1),(2) can be derived as the Euler-Lagrange equations for the Lagrangian density (10) and, therefore, do provide an extremum for the action $S = \int\int_D L d\mathbf{x} dt$, where the variations $\delta\mathbf{A}$ and $\boldsymbol{\chi}$ vanish at the boundary ∂D of the integration domain D, $(\delta\mathbf{A}, \boldsymbol{\chi})\ |_{\partial D} = 0$.

4. We restrict ourselves to looking for the Lie-Bäcklund symmetries of the 1st order, i.e., for the infinitesimal transformations which depend on $\mathbf{A}$, α^j and their first derivatives over t and $\mathbf{x}$ and leave Eqs.(1)–(4) unchanged. As a result of applying a standard algorithm of finding the

admitted Lie-Bäcklund symmetries [7], one arrives at:

$$\boldsymbol{\xi} = -(b_0 + b_1 t)\mathbf{e}_i \partial_t \alpha^i + \mathbf{c}_1 + \mathbf{c}_2 \times \mathbf{x} + \xi^{*i}(\alpha^j)\mathbf{e}_i,$$
$$\delta\mathbf{A} = (b_0 + b_1 t)\partial_t \mathbf{A} + b_1 \mathbf{A} + \mathbf{c}_2 \times \mathbf{A} - ((\mathbf{c}_1 + \mathbf{c}_2 \times \mathbf{x})\boldsymbol{\nabla})\,\mathbf{A} + \boldsymbol{\nabla} F, \qquad (11)$$

where b_0, b_1 are arbitrary constants, F is a gauge function, which is still arbitrary, the components of $\mathbf{c}_1$ and $\mathbf{c}_2$ are constants, which should satisfy the additional equations: $\mathbf{c}_1 \boldsymbol{\nabla}\rho = (\mathbf{c}_2 \times \mathbf{x})\boldsymbol{\nabla}\rho = 0$. The functions $\xi^{*i}(\alpha^j)$ depend on the equation of state of the electron fluid. If $\boldsymbol{\nabla} T \times \boldsymbol{\nabla}\eta \equiv 0$, then ξ^{*i}-s are arbitrary functions. When $\boldsymbol{\nabla} T \times \boldsymbol{\nabla}\eta \neq 0$ anywhere, but $T = T_1(\rho)T_2(\alpha^1) + T_3(\alpha^1)$, ξ^{*1} is defined by:

$$\xi^{*1} = -2b_1 \frac{\int T_2 \frac{d\eta}{d\alpha^1} d\alpha^1}{T_2 \frac{d\eta}{d\alpha^1}} + \frac{b_2}{T_2 \frac{d\eta}{d\alpha^1}}, \qquad \left(T_2 \frac{d\eta}{d\alpha^1} \neq 0\right), \qquad (12)$$

where b_2 is arbitrary constant, ξ^{*2}, ξ^{*3} are arbitrary functions, and α^1 is chosen to provide $\eta = \eta(\alpha^1)$. Finally, if $\boldsymbol{\nabla} T \times \boldsymbol{\nabla}\eta \neq 0,\ T \neq T_1(\rho)T_2(\alpha^1) + T_3(\alpha^1)$, then $b_1 = 0,\ \xi^{*1} = 0$ and ξ^{*2}, ξ^{*3} are arbitrary functions. The Lie-Bäcklund symmetries of EMHD equations can be divided into two classes. The symmetries of the first class, defined by the coefficients $b_0, b_1, b_2, \mathbf{c}_1, \mathbf{c}_2$ are equivalent to the Lie point transformations (time and space translations, generalized rotations, scalings) and could be more easily found from Eqs.(1)–(4) written in terms of the physical variables. The symmetries of the second class described by ξ^{*i} are the so-called relabeling symmetries, and they do not change the physical variables at all ($\delta\mathbf{v} = \delta\mathbf{B} = 0,\ \delta\eta = \delta T = 0$).

5. Any Lie-Bäcklund transformation is a variational symmetry of the action S if and only if for any $\mathbf{A}(t, \mathbf{x})$, $\alpha^i(t, \mathbf{x})$, which satisfy EMHD equations, there exist the functions $\Lambda^0, \boldsymbol{\Lambda}$ such that

$$\delta L + \partial_t \Lambda^0 + \boldsymbol{\nabla} \cdot \boldsymbol{\Lambda} = 0. \qquad (13)$$

According to the Noether's theorem in the Boyer's formulation [8], the variational symmetry, which corresponds to Eq.(13), generates the conservation law

$$\partial_t \left\{\mathbf{u}(\boldsymbol{\nabla} \times \boldsymbol{\chi}) + \Lambda^0\right\} + \boldsymbol{\nabla} \cdot \Big\{\mathbf{v}\,(\mathbf{u}(\boldsymbol{\nabla} \times \boldsymbol{\chi})) + \mathbf{B} \times \delta\mathbf{A}$$
$$+\boldsymbol{\Lambda} + (\partial_t \mathbf{u} - \mathbf{v} \times \boldsymbol{\Omega} + \boldsymbol{\nabla}(\mathbf{v}\mathbf{u}) - T\boldsymbol{\nabla}\eta) \times \boldsymbol{\chi}\Big\} = 0. \qquad (14)$$

It is easy to check that symmetry related to b_0 (time translation) leads as usually to the energy conservation, while the space translations ($\mathbf{c}_1$) and rotations ($\mathbf{c}_2$) generate the conservation laws of momentum and angular momentum. The relabeling symmetry ξ^{*i} is a variational symmetry of the Lagrangian density L if

$$\boldsymbol{\xi}^* \boldsymbol{\nabla}\eta = 0. \qquad (15)$$

The relabeling symmetry generates a local conservation law of the form:

$$\partial_t(\rho \mathbf{u}\boldsymbol{\xi}^*) + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}(\mathbf{u}\boldsymbol{\xi}^*)) = 0. \qquad (16)$$

In deriving this equation from (14) it was taken into account that

$$\partial_t \mathbf{u} - T\boldsymbol{\nabla}\eta - \mathbf{v} \times \boldsymbol{\Omega} = \boldsymbol{\nabla}\psi, \qquad (17)$$

as it follows from Eq.(1), and a special gauge $\psi = -\mathbf{u}\mathbf{v}$ was used.

>From Eq.(15) it follows that in the isentropic case $\boldsymbol{\nabla}\eta \equiv 0$ any relabeling symmetry is a

variational symmetry, and the conservation law (16) is integrated by any vector of the form:

$$\mathbf{u} = u_i^*(\alpha^j)\mathbf{e}^i, \tag{18}$$

which is a *general* solution of the equation of the motion (1). In particular case $\boldsymbol{\xi}^* = \boldsymbol{\Omega}/\rho$, Eq.(16) gives

$$\partial_t \int (\mathbf{u}\boldsymbol{\Omega}) d\mathbf{x} = 0, \tag{19}$$

and an infinite set of Casimirs can be found by choosing $\boldsymbol{\xi}^* = \boldsymbol{\Omega} F(\alpha^j)/\rho$:

$$\int (\mathbf{u}\boldsymbol{\Omega}) F\left(\frac{\mathbf{u}\boldsymbol{\Omega}}{\rho}, \eta, \frac{\boldsymbol{\Omega}}{\rho}\boldsymbol{\nabla}\left(\frac{\mathbf{u}\boldsymbol{\Omega}}{\rho}\right), ...\right) d\mathbf{x}. \tag{20}$$

The similar set of Casimirs can be also organized for *any relabeling*

$$\boldsymbol{\xi}^* = \xi^{*n}(\{\alpha^i\})\mathbf{e}_n = \frac{\epsilon^{njk}}{R}\frac{\partial \chi_k^*}{\partial \alpha^j}\mathbf{e}_n\ , \ \text{where } \chi_k^* = \chi_k^*(\{\alpha_i\}) \ - \text{arbitrary functions} .$$

In the case $\boldsymbol{\nabla}\eta \neq 0$ the relabeling symmetry is a variational symmetry only if $\xi^{*1} = 0$. Then the conservation law (16) allows to integrate two components of $\mathbf{u}$: $u_{2,3} = u_{2,3}(\alpha^i)$, and this fact means an important simplification of EMHD in comparison with ideal MHD, where such an integration is impossible for a nondegenerated state [6].

6. Although only the Lie-Bäcklund symmetries of the first order are found in this paper, linear dependence of ξ^i and $\delta\mathbf{A}$ on the derivatives of α^j and $\mathbf{A}$ says, as experience shows, in favour of a conjecture, that there should be no additional Lie-Bäcklund symmetries of the higher orders, and *all* Lie-Bäcklund symmetries of EMHD could be reduced to the calculated ones. In turn, this fact gives a ground to assume that *all* the conservation laws of EMHD are known. The above said together with a knowledge, that the system is a Hamiltonian one and admits a canonical representation, constitutes a solid ground for a subsequent stability analysis for the stationary solutions of EMHD equations.

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