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Magnetohydrodynamic Equilibria of a Compressible Plasma with Mass Flow in an Axisymmetric Tokamak

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The reduction of the vector equations. As starting point we consider the following set of well-known stationary, non-linear single-fluid MHD equations, which describe the macroscopic dynamics of ideal compressible plasma with mass flow

$$\rho\left(\vec{v}\cdot\nabla\right)\vec{v} = -\nabla p + (1/c)\,\vec{j}\times\vec{B}, \quad \nabla\times\vec{B} = (4\pi/c)\,\vec{j}, \quad \nabla\times\vec{E} = \nabla\cdot\vec{B} = 0, \qquad (1)$$

where ρ, $\vec{v}$, and p are, respectively, plasma mass density, velocity and pressure, $\vec{E}$and $\vec{B}$ are the electric and magnetic fields, $\vec{j}$ is the current density.

In order to close this set of equations in addition one has to apply ideal Ohm's law and equation of plasma state

$$\vec{E} + (1/c)\,\vec{v}\times\vec{B} = 0, \quad \vec{E} = -\nabla\Phi, \quad \left(\vec{v}\cdot\nabla\right)S = 0, \qquad (2)$$

where $S = p/\rho^{\gamma}$ is entropy.

Axially symmetric solutions of the system of Eqs. (1)-(2) are considered in arbitrary co-ordinates (x^1, x^2, x^3) [1]. According to our assumption we put $\partial/\partial x^3 = 0$ and introduce the velocity and magnetic fields in the form

$$\vec{v} = \frac{1}{\rho}\left(\nabla\varphi\times\vec{e}^{\,3}\right) + v^3\vec{e}_3, \quad \vec{B} = \left(\nabla\psi\times\vec{e}^{\,3}\right) + B^3\vec{e}_3, \qquad (3)$$

where the superscripts and subscripts indicate contravariant and covariant components of a vector, respectively.

By taking into account the representation (3), the system of Eqs. (1) - (3) can after some manipulations be reduced to the following scalar partial differential equations

$$\begin{aligned}
&\{\varphi,\psi\} = \{\phi,\psi\} = 0, \quad \frac{1}{4\pi}\{g_{33}B^3,\psi\} = \{g_{33}v^3,\Phi\},\\
&\left\{v^3 - \frac{d\Phi}{d\psi}\frac{B^3}{\rho},\psi\right\} = 0, \quad \{p/\rho^{\gamma},\psi\} = 0,\\
&\left\{\frac{\vec{v}^2}{2} + S(\psi)\frac{\gamma}{\gamma-1}\rho^{\gamma-1} - g_{33}v^3c\frac{d\Phi}{d\psi},\psi\right\} = 0,\\
&\Delta\psi\nabla\psi + B^3\nabla\left(g_{33}B^3\right) - 4\pi\nabla\varphi\left[\frac{\Delta\varphi}{\rho} + \frac{1}{g_{33}}\nabla\left(\frac{1}{\rho}\right)\cdot\nabla\varphi\right]\\
&\quad - 4\pi\rho v^3\nabla\left(g_{33}v^3\right) + 4\pi\nabla P - 2\pi|\vec{v}|^2\nabla\rho = 0,
\end{aligned} \qquad (4)$$

where the expressions $P, \{\cdots\}$ and Δ in (4) are given by

$$\begin{aligned}
&P = p + \rho\vec{v}^2/2, \quad \{X,Y\} \equiv \frac{1}{\sqrt{g}}\left(\frac{\partial X}{\partial x^1}\frac{\partial Y}{\partial x^2} - \frac{\partial X}{\partial x^2}\frac{\partial Y}{\partial x^1}\right),\\
&\Delta X \equiv \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^1}\left(\frac{g_{22}}{\sqrt{g}}\frac{\partial X}{\partial x^1} - \frac{g_{12}}{\sqrt{g}}\frac{\partial X}{\partial x^2}\right) + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}\left(\frac{g_{11}}{\sqrt{g}}\frac{\partial X}{\partial x^2} - \frac{g_{12}}{\sqrt{g}}\frac{\partial X}{\partial x^1}\right),
\end{aligned} \qquad (5)$$

g_{ik} are the coefficients of metric tensor, $g = \det g_{ik}$.

Eqs. (4) describe the general structure of the magnetic, electric and the velocity fields, and of the pressure and density profiles for stationary axisymmetric toroidal dynamics, where the velocity and magnetic fields are assumed to be in general non-parallel.

The modified Grad – Shafranov equation. With the help of Eqs. (4) and (5) one may easily find the following useful expressions required in the following investigations

$$\begin{aligned} B^3 &= \left[I/g_{33} + 4\pi c\,(d\Phi/d\psi)\,(d\varphi/d\psi)\right] / \left[1 - (4\pi/\rho)\,(d\varphi/d\psi)^2\right], \\ v^3 &= \left[(I/g_{33})\,(d\varphi/d\psi)\,(1/\rho) + c\,(d\Phi/d\psi)\right] / \left[1 - (4\pi/\rho)\,(d\varphi/d\psi)^2\right], \end{aligned} \tag{6}$$

where the poloidal current $I = I(\psi)$ is a surface function. In general, the pressure and density are no flux function.

By taking into account the relations

$$\begin{aligned} &\nabla\varphi = (d\varphi/d\psi)\,\nabla\psi, \quad \Delta\varphi = (d\varphi/d\psi)\,\Delta\psi + \left(d^2\varphi/d\psi^2\right)|\nabla\psi|^2, \\ &|\nabla\psi|^2 = \frac{g_{11}}{g}\left(\frac{\partial\psi}{\partial x^2}\right)^2 - \frac{2g_{12}}{g}\left(\frac{\partial\psi}{\partial x^1}\right)\left(\frac{\partial\psi}{\partial x^2}\right) + \frac{g_{22}}{g}\left(\frac{\partial\psi}{\partial x^1}\right)^2 \end{aligned} \tag{7}$$

we then infer that Eqs. (4) lead to the modified Grad – Shafranov equation for compressible plasma with mass flow in the form

$$\begin{aligned} &g_{33}\frac{\Delta\psi}{4\pi}\left[1 - \frac{4\pi}{\rho}\left(\frac{d\varphi}{d\psi}\right)^2\right] - \frac{g_{33}}{2\rho}\frac{d}{d\psi}\left[\left(\frac{d\varphi}{d\psi}\right)^2\right](\nabla\psi)^2 + \left(\frac{d\varphi}{d\psi}\right)^2\nabla\left(\frac{1}{\rho}\right)\cdot\nabla\psi + \\ &+ \frac{g_{33}B^3}{4\pi}\frac{dI}{d\psi} + (g_{33})^2 v^3 B^3\frac{d^2\varphi}{d\psi^2} + \rho\,(g_{33})^2 v^3 c\frac{d^2\varphi}{d\psi^2} + g_{33}\rho\frac{dH}{d\psi} - \rho^\gamma\frac{dS}{d\psi}\frac{g_{33}}{\gamma - 1} = 0 \quad , \\ &S = p/\rho^\gamma = S(\psi), \quad H = \frac{\vec{v}^2}{2} + S(\psi)\frac{\gamma}{\gamma - 1}\rho^{\gamma - 1} - g_{33}v^3 c\frac{d\Phi}{d\psi} = H(\psi). \end{aligned} \tag{8}$$

Based on cylindrical co-ordinates $(x^1 = R, x^2 = z, x^3 = \varphi)$, when $\psi = \psi(R, z)$, we derive with the help of (6), (7), (8) the Kerner – Tokuda equation [2] for a compressible plasma with mass flow

$$\begin{aligned} &\mathrm{div}\left\{\left[1 - \frac{4\pi}{\rho}\left(\frac{d\varphi}{d\psi}\right)^2\right]\frac{\nabla\psi}{R^2}\right\} + 4\pi\vec{v}\cdot\vec{B}\frac{d^2\varphi}{d\psi^2} + \frac{4\pi}{R^2}\frac{\left[I + 4\pi R^2 c\frac{d\Phi}{d\psi}\frac{d\varphi}{d\psi}\right]}{\left[1 - \frac{4\pi}{\rho}\left(\frac{d\varphi}{d\psi}\right)^2\right]}\frac{dI}{d\psi} + 4\pi\rho\frac{dH}{d\psi} \\ &- 4\pi\rho^\gamma\frac{dS}{d\psi}\frac{1}{\gamma - 1} + \frac{4\pi\rho\left(\frac{I}{\rho}\frac{d\varphi}{d\psi} + R^2 c\frac{d\Phi}{d\varphi}\right)}{1 - \frac{4\pi}{\rho}\left(\frac{d\varphi}{d\psi}\right)^2}c\frac{d^2\Phi}{d\psi^2} = 0. \end{aligned} \tag{9}$$

In the static limit $(d\varphi/d\psi = d\Phi/d\psi = 0)$ Eq. (9) reduces to the Grad – Shafranov equation for the potential $\psi(R, z)$. Eq. (9) has an infinite number of classes of mathematical solutions each corresponding to a particular set of the arbitrary functions of $\psi(a, \theta)$. We now discuss solutions of Eqs.(8) and (9) in the context of plasma equilibrium in tokamak with mass flow.

Equilibrium with nested magnetic surfaces. Following [3], we assume that the magnetic flux surfaces consist of a family of closed nested-in toroidal surfaces with "on-average" circular cross-section so that we can introduce a radial coordinate a, which coincides with the average radius of the magnetic surfaces. We also suppose that magnetic surfaces wrapped around a single magnetic axis, which is shifted relatively to the geometric axis by an amount ξ. We introduce on the magnetic surfaces $a = \text{const}$ the poloidal (θ) and toroidal ϕ angles as independent variables. In these "quasi-toroidal" co-ordinates (a, θ, ϕ) a magnetic field can be represented in the form

$$B^i = \frac{1}{2\pi\sqrt{g}} \left(0, \chi', \Theta'\right), \tag{10}$$

where χ and Θ are the poloidal and toroidal magnetic fluxes, $(\cdots)' \equiv \partial\,(\cdots)/\partial a$.Thus the representation (10) of the magnetic field is formally obtained from the one of Eq. (3) by transformation

$$B^3\,(a,\theta) \Rightarrow (1/2\pi\sqrt{g})\,\Theta', \quad \psi\,(a,\theta) \Rightarrow \psi\,(a) = -\,(1/2\pi)\,\chi\,(a)\,.$$

Since the potential ψ only depends on the radial co-ordinate a, immediately leading to the following relations for the physical variables

$$\varphi = \varphi\,(a)\,, \Phi = \Phi\,(a)\,, I = I\,(a)\,, S = S\,(a)\,, H = H\,(a)\,,\ v^1 = B^1 = 0,\ v^2 = (d\varphi/d\psi)\,B^2,$$

$$\Delta\psi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial a}\left(\frac{g_{22}}{\sqrt{g}}\frac{\partial\psi}{\partial a}\right) - \frac{1}{\sqrt{g}}\frac{\partial}{\partial\theta}\left(\frac{g_{12}}{\sqrt{g}}\frac{\partial\psi}{\partial a}\right), \quad |\nabla\psi|^2 = \frac{g_{22}}{g}\left(\frac{\partial\psi}{\partial a}\right)^2 \quad .$$

Stationary toroidal equilibria, described by Eq. (8), can be investigated by means of an expansion with respect to the inverse aspect ratio $\varepsilon = a/R$. Following [3], we obtain with the help of Eq. (8) by neglecting terms of the order $O\,(\varepsilon^3)$ the non-zero components of the metric tensor g_{ik} in the form:

$$\begin{aligned} g_{11} &= 1 + 2\xi'\cos\theta, g_{12} = \left(a^2\lambda' - a\xi'\right)\sin\theta, g_{22} = a^2 + 2\lambda a\cos\theta, \\ g_{33} &= (R - \xi + a\lambda/2 - a\cos\theta)^2\,, \quad \sqrt{g} = aR\,(1 + (\xi' + \lambda - ka)\cos\theta)\,, \end{aligned} \tag{11}$$

where λ is the parameter of "strightforward" magnetic force lines (see details in [3]).

We employ the first equation (6) to in order to find an expression for the unknown parameter λ. Under the conventional tokamak ordering (the superscript "0" denotes the cylindrical terms)

$$q \approx 1, B_\theta^{(0)}\Big/B_\phi^{(0)} \approx \varepsilon, \quad p^{(0)}\Big/B_\phi^{(0)}, \rho\left(v_\phi^{(0)}\right)^2\Big/\left(B_\phi^{(0)}\right)^2, \rho\left(v_\theta^{(0)}\right)^2\Big/\left(B_\theta^{(0)}\right)^2 \approx \varepsilon^2$$

the Eq. (6) gives the following well-known expression for λ

$$\lambda = -\xi' - ka \quad . \tag{12}$$

The components of the metric tensor (11) together with (12) fully describe the geometrical properties of the equilibrium with mass flow.

In the leading order $O\,(\varepsilon^2)$ Eq. (9) describes the stationary cylindrical equilibrium

$$\frac{d}{da}\left[p^{(0)} + \frac{(B_\theta^{(0)})^2 + (B_\phi)^2}{8\pi}\right] + \frac{1}{4\pi a}\left[(B_\theta^{(0)})^2 - 4\pi\rho\left(v_\theta^{(0)}\right)^2\right] = 0, \tag{13}$$

here

$$B_\phi = B_\phi^{(0)} + B_\phi^{(2)}, \quad B_\phi^{(0)} = \frac{\Theta}{2\pi a}, B_\phi^{(2)} = -B_\phi^{(0)}\left(k\xi + ka\xi'/2 + k^2a^2/2\right), B_\theta^{(0)} = \frac{I}{2\pi R}\frac{d\chi}{da}.$$

In the next order ($\approx \varepsilon^3$), Eq.(9) describes the first – order toroidal correction to the cylindrical equilibrium (13) leading to

$$\frac{1}{a}\frac{d}{da}\left(a\xi' B_\theta^{(0)2}\right) = \frac{B_\theta^{(0)2}}{R} - \frac{8\pi a}{R}\frac{d}{da}\left(p^{(0)} + \rho v_\phi^{(0)2}/2\right), \tag{14}$$

with the relations

$$v_\phi^{(0)} = \frac{v_\theta^{(0)}}{B_\theta^{(0)}}B_s^{(0)} + \frac{cE_a^{(0)}}{B_\theta^{(0)}}, \qquad v_\theta^{(0)} = \frac{d\varphi}{d\psi}B_\theta^{(0)}.$$

If the plasma is surrounded by a perfectly conducting rigid wall of radius $a = a_0$, then we have $\xi\,(a = a_0) = 0$. With the help of this boundary condition, we obtain from Eq.(14) the following simple expression for the Shafranov shift ξ:

$$\xi = -\frac{1}{R}\int_a^{a_0}\frac{da'}{a' B_\theta^{(0)\,2}}\int_0^{a'} a'' B_\theta^{(0)\,2} da'' + \frac{8\pi}{R}\int_a^{a_0}\frac{da'}{a' B_\theta^{(0)\,2}}\int_0^{a'} a''^2\left(p^{(0)} + \frac{\rho v_\phi^{(0)\,2}}{2}\right)da''.$$

In the case of $d\varphi/d\psi = 0$, Eqs. (13) and (14) describe an equilibrium with purely toroidal rotation and in the case $d\Phi/d\psi = 0$ — one with purely parallel rotation.

On conclusion we note, that from the conditions $S = S\,(a)\,, H = H\,(a)$,it follows, that in the presence of flow the magnetic surfaces do not coincide with the one of constant pressure and density

$$p = p^{(0)}\,(a) + p_{(a)}^{(1)}\cos\theta, \quad \rho = \rho^{(0)}\,(a) + \rho_{(a)}^{(1)}\cos\theta,$$

$$p^{(0)} = S\,(a)\,\rho^{(0)\gamma}, \quad p^{(1)} = c_S^2\rho^{(1)}, \quad c_S^2 = \frac{\gamma p^{(0)}}{\rho^{(0)}},$$

$$\rho^{(0)} = \left\{\frac{\gamma - 1}{S\,(a)\,\gamma}\left[H\,(a) - \frac{1}{2}\left(v_\theta^{(0)}\frac{B_s^{(0)}}{B_\theta^{(0)}}\right)^2 + \frac{1}{2}\left(\frac{cE_r^{(0)}}{B_\theta^{(0)}}\right)^2\right]\right\}^{\frac{1}{\gamma-1}},$$

$$\rho^{(1)} = \rho^{(0)}\frac{a}{R}\frac{\left(\frac{cE_r^{(0)}}{B_\theta^{(0)}}\right)^2 + \left(v_\theta^{(0)}\frac{B_s^{(0)}}{B_\theta^{(0)}}\right)^2}{\left(v_\theta^{(0)}\frac{B_s^{(0)}}{B_\theta^{(0)}}\right)^2 - c_s^2}.$$

Now the pressure and density consist of two parts: the terms $p^0\,(a)$ and $\rho^0\,(a)$, which are constant on the magnetic surfaces, and its deviations $p^{(1)}\,(a)\cos\theta$ and $\rho^{(1)}\,(a)\cos\theta$ arising from a finite plasma flow.

References

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