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Hermitian moment method for non-maxwellian distributions during ICRH

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Recent analytical studies of neoclassical plasma poloidal rotation during ICRH [1,2] were based on a method which was not totally internally self-consistent since the local equilibrium distribution function was given in the form of a bi-maxwellian while the transport calculations assumed an isotropic temperature. It is therefore of interest to develop a transport theory that allows for anisotropic temperatures. As a model system for this study, we consider a kinetic equation sum of an ion-ion collision term and a quasi-linear heating term describing the interaction of the ions with the RF in the ion cyclotron layer. A usual way to solve this kind of kinetic equation is to decompose the ion distribution function into a maxwellian "bulk" distribution function and a "tail" distribution function and to consider the bulk-tail ion collisions. We avoid this quite arbitrary decomposition of the distribution function by using the Landau collision operator instead of the Fokker-Planck collision operator and by using a moment method starting from the set of irreducible Hermite polynomials already used in another context in [3]. The reference local equilibrium distribution function is then represented by a product of a bi-maxwellian in $\{v_{//}, v_{\perp}\}$, which takes into account the anisotropy in temperature, with a function of position, velocity and time. The latter function is expanded in the polynomials. With this method, the ion distribution function valid in the whole velocity space is obtained. An equation for the parallel component of the temperature in terms of the perpendicular component of the temperature and of the heating power is also derived. This equation, in general too difficult to be solved analytically, can be treated numerically. In our formalism one component of the "anisotropic" temperature remains a free parameter. This is justified by the fact that in absence of heating the "isotropic" temperature is not determined by the local equilibrium kinetic equation but rather by the transport processes [4].

Basic equations

A two component plasma is considered where the ion distribution function f satisfies the kinetic equation $\mathrm{K(f,f)+Q(f)=0}$ where $\mathrm{K(f,f)}$ is the Landau ion-ion collision operator and $\mathrm{Q(f)}$ the quasi-linear heating operator (ion-electron collisions are neglected). We assume that the ion distribution function is well represented by an expansion in tensorial Hermite polynomials: $f(q,c) = A(q)\,\Phi_0(c)\,(1+\kappa(q,c))$ where $1+\kappa(q,c) = \sum_{n,ab}\Phi^{(n)}_{ab}(q)\,H^{(n)}_{ab}(c)$ with $\Phi^{(0)} = 1$ and $H^{(n)}_{ab}$ are the tensorial Hermite polynomials, $A(q)$ is the normalization, and $\Phi_0(c)$ is a maxwellian. The moments of the distribution function are denoted $\Phi^{(n)}_{ab}$. Here, we assume a tensorial relationship between the phase space variable c and the ion velocity v, thus allowing for an anisotropic temperature: $c = \mu(q)\cdot(v-u)$. The unknowns in the moment method are the components of the tensor $\mu(q)$, which for simplicity is assumed to be diagonal $\mu = \sqrt{\mu_{//}}\,bb + \sqrt{\mu_\perp}\,(1-bb)$. The vector b is a unit vector taken along the main magnetic field. Since the maxwellian $\Phi_0(c)$ is the solution of the zeroth order kinetic equation $K(\Phi_0,\Phi_0) = 0$, the usual five collisional invariants (the number of particles n, the mean velocity u and the internal energy or v^2) already define five of the moments of the distribution function. That is $\Phi^{(0)} = 1$, $\Phi^{(1)}_r = 0$ and $\left(\frac{1}{\mu_{//}} - \frac{1}{\mu_\perp}\right)\sqrt{2}\left\{\frac{2}{3}\Phi^{(2)} : bb - \frac{1}{3}\Phi^{(2)} : (1-bb)\right\} + \left(\frac{1}{\mu_{//}} + \frac{2}{\mu_\perp}\right)\frac{\sqrt{6}}{3}\Phi^{(2)} = 0$. In the case of an isotropic temperature $\mu_{//} = \mu_\perp$ the latter relation reduces to $\Phi^{(2)} = 0$, as expected. An additional relation $\Phi^{(2)} : bb + \Phi^{(2)} : (1-bb) = 0$ results from the traceless form of the second rank tensors. Thus of the three moments of the distribution function $\left\{\Phi^{(2)}, \Phi^{(2)} : bb, \Phi^{(2)} : (1-bb)\right\}$ only one, $\Phi^{(2)} : bb$ say, is an unknown. Since we do not consider any current drive in the model, there is a decoupling of the scalar and odd tensorial moment equations and the vectorial and even tensorial moments. Thus for the sake of simplicity of this presentation, we do not write down the vectorial and even moments of the distribution function. Therefore, in the 13-moment approximation, there remain two functions to be determined: $\Phi^{(2)} : bb$ and $\mu_\perp$ which will be obtained in terms of the heating power and of the temperature $T = \frac{1}{3}T_{//} + \frac{2}{3}T_\perp = \frac{1}{3}\frac{m}{\mu_{//}} + \frac{2}{3}\frac{m}{\mu_\perp}$. The ion distribution function, as said before, reduces to

$$f = A\,\Phi_0(c)\left\{H^{(0)}(c) - \left(3H^{(2)}(c) - 2H^{(2)}(c) : bb\right)\Phi^{(2)}(q) : bb\right\}$$

where the irreducible tensorial Hermite polynomials are $H^{(0)} = 1$, $H^{(2)} = \frac{1}{\sqrt{6}}\left(c^2 - 3\right)$, $H^{(2)}_{ij} = \frac{1}{\sqrt{2}}\left(c_i\,c_j - \frac{1}{3}c^2\,\delta_{ij}\right)$. Obviously, in absence of heating this distribution function must reduce to $f = A\,\Phi_0(c)$ and $\mu_{//} = \mu_\perp$.

The moment equations

The moment method leads to two equations composed of collisional and heating matrix elements: $\left\langle H^{(2)} \middle| K \middle| f,f \right\rangle + \left\langle H^{(2)} \middle| Q \middle| f \right\rangle = 0$ and $\left\langle H^{(2)} : bb \middle| K \middle| f,f \right\rangle + \left\langle H^{(2)} : bb \middle| Q \middle| f \right\rangle = 0$ where we have introduced the notation $\langle x|y|z \rangle = \int d^3c\, (x\, y\, z)$. The heating operator is defined as $Q(f) = Q_0 \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} v_\perp J_{n-1}^2 \left(\frac{v_\perp k_\perp}{\Omega} \right) \frac{\partial}{\partial v_\perp} f$ whereas the collision operator is

$K(f,f) = \frac{2\pi e_i^4}{m_i^2} \ln\Lambda \int dv_2 \frac{\partial}{\partial v_{1r}} G_{rs}(g) \left(\frac{\partial}{\partial v_{1s}} - \frac{\partial}{\partial v_{2s}} \right) f(1) f(2)$ where $G_{rs}(a) = \frac{a^2 \delta_{rs} - a_r a_s}{a^3}$ is the Landau tensor and $g = v_1 - v_2$. The coulomb logarithm $\ln\Lambda$ is $\ln\Lambda = \ln \frac{\frac{3}{2}(T_e + T_i)\lambda_D}{Ze^2}$.

These moment equations are usually linearized so that $\langle \psi | K | f,f \rangle_{lin} \equiv \langle \psi | K | (f, A\Phi_0) + (A\Phi_0, f) \rangle$. In this regime, one of the two equations determines the moment of the distribution function

$\Phi^{(2)} : bb = \frac{\left\langle H^{(2)} \middle| K' \middle| H^{(0)} \right\rangle}{\left(3\left\langle H^{(2)} \middle| K' \middle| H^{(2)} \right\rangle - 2\left\langle H^{(2)} \middle| K' \middle| H^{(2)} : bb \right\rangle \right)}$ while the other leads to a kind of compatibility equation which determines the ratio of the “perpendicular temperature” to the “parallel temperature” in terms of the heating power:

$$\frac{\left\langle H^{(2)} \middle| K' \middle| H^{(0)} \right\rangle}{\left(3\left\langle H^{(2)} \middle| K' \middle| H^{(2)} \right\rangle - 2\left\langle H^{(2)} \middle| K' \middle| H^{(2)} : bb \right\rangle \right)} = \frac{\left\langle H^{(2)} : bb \middle| K' \middle| H^{(0)} \right\rangle}{\left(3\left\langle H^{(2)} : bb \middle| K' \middle| H^{(2)} \right\rangle - 2\left\langle H^{(2)} : bb \middle| K' \middle| H^{(2)} : bb \right\rangle \right)}$$

All the integrals involved here can be performed analytically. For instance, the heating matrix elements are all similar in form to $\left\langle H^{(2)} \middle| Q \middle| H^{(2)} \right\rangle = n \frac{4}{6} Q_n \left(\frac{W_2}{W_1} - 4 \right)$ where

$Q_n = \pi \frac{A}{n} \mu_\perp Q\, U_0 W_1$, $U_s = \mu_{//}^s \left(-2 \frac{\partial}{\partial \mu_{//}} \right)^s U_0$ and $W_s = \mu_\perp^s \left(-2 \frac{\partial}{\partial \mu_\perp} \right)^s W_0$ with $U_0 = \frac{1}{2} \sqrt{\frac{2\pi}{\mu_{//}}}$

and $W_0 = 2 \left(\frac{\Omega}{k_\perp} \right)^2 x\, I_{n-1}(x)\, e^{-x}$. Here $I_{n-1}(x)$ is the modified Bessel function and x, a dimensionless variable $x = \frac{1}{\mu_\perp} \left(\frac{k_\perp}{\Omega} \right)^2$. The collisional matrix elements are much more involved. Indeed, they have a form similar to the matrix element $\left\langle H^{(2)} \middle| K \middle| H^{(0)}, H^{(0)} \right\rangle = \frac{n}{\sqrt{6}} C \left\{ 2(S + 4T)(\mu_{//} + 2\mu_\perp) - 4T\left(\sqrt{\mu_{//}} - \sqrt{\mu_\perp} \right)^2 \right\}$ where S and T are complicated functions of the plasma parameters (i.e. also of $\{\mu_{//},\mu_{\perp}\}$) and $C = \frac{8}{3}\sqrt{2\pi}\frac{e^4}{m^2} n \ln\Lambda$. Only the study of the compatibility equation requires a numerical approach. The non-linear analysis proceeds exactly the same way except that we now have to deal with a quadratic equation for $\Phi^{(2)} : bb$ which therefore may have two real solutions. The present discussion concerned only a part of the distribution function related to the temperature and the pressure tensor. The part not discussed here, i.e. corresponding to the vectorial and the odd tensorial moments of the distribution function is also important since it relates to the geometry of the distribution in the $\{v_{//}, v_{\perp}\}$ plane with the typical rabbit ears observed in the case of ion cyclotron heating.

Conclusions

We have shown that the hermitian moment method used in the neoclassical transport theory (see Ref. [3]) can be generalized and adapted to plasmas that have strong anisotropic local equilibrium distribution functions induced by RF heating. Furthermore, as we have also demonstrated, this method allows the use of the complete distribution function instead of the coupled distributions for the bulk and the tail of the heated species, which is a major drawback for the development of a transport analysis. However, the Hermitian moment method has also its limitations, namely the dependence of the distribution function in the velocity is by definition polynomial, a prescription that might not be followed by the observations. The fact that, in the present analysis, the polynomials are in "c" and not in "v" (however here related tensorially) opens a door for more complex, eventually non linear, relations, between the two quantities. This freedom could eventually be used to obtain distribution functions closer to those expected for a RF modified local equilibrium distribution function.

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