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Modulational instability and localized excitations in dusty plasma crystals

Ioannis Kourakis

U.L.B. - Université Libre de Bruxelles

Physique Statistique et Plasmas, Association Euratom - Etat Belge

C.P. 231, boulevard du Triomphe, B-1050 Brussels, Belgium

email: ikouraki@ulb.ac.be

Abstract

We study the occurrence of modulational instability on waves ('phonons') propagating in a single layer dusty plasma crystal. This instability mechanism, related to the intrinsic nonlinearities of the sheath electric field, is expected to occur under certain conditions, possibly leading to the formation of localized excitations. The instability rate and threshold are obtained in terms of the system dispersion laws.

Modulational instability (MI) is a well-known mechanism of energy localization occurring during wave propagation in nonlinear dispersive media. It has been thoroughly studied in the past, mostly in one-dimensional solid state systems, where nonlinearities of the substrate potential and/or particle coupling may be seen to destabilize waves and possibly lead to localized excitations (solitary waves). However, to our knowledge, no such study has been carried out in the case of dusty plasma (DP) crystals [1] i.e. lattices of (strongly-coupled) dust grains which have recently received increasing interest, due to their occurrence in real plasmas and the novel physics involved in their description.

It has been proved that DP crystals support low-frequency optical-mode-like oscillations in both transverse and longitudinal directions [2]. Focusing on the former and summarizing previous results, let us recall that transverse motion in a crystal of charged dust grains (mass M, charge Q, lattice constant r_0) obeys an equation of the form:

$$M\,\frac{d^2\delta z_n}{dt^2} = M\,\omega_0^2\,(2\,\delta z_n - \,\delta z_{n-1} - \,\delta z_{n+1}) + F_e - Mg \qquad (1)$$

where $\delta z_n = z_n - z_0$ denotes the small displacement of the $n-$th grain around the equilibrium position z_0, in the transverse direction $(z-)$, propagating in the longitudinal $(x-)$ direction; ω_0 is the DP transverse oscillation 'eigenfrequency':

$$\omega_0^2 = \frac{Q^2}{Mr_0^3}\,(1+\frac{r_0}{\lambda_D})\,e^{-r_0/\lambda_D} \qquad (2)$$

resulting from the Debye interaction potential. Solving Poisson's equation, one obtains the electric field, which is due to the sheath potential *and* (in a more complete picture) to the wake potential generated by supersonic ion flow towards the electrode [3]. The total field $E(z)$, actually a *nonlinear* function of z, can be developed around the equilibrium position; the electric force therefore reads:

$$F_e(z) \approx F_e(z_0) + \gamma_{(1)}\, \delta z + \gamma_{(2)}\, (\delta z)^2 + \gamma_{(3)}\, (\delta z)^3 + \mathcal{O}((\delta z)^4)$$

where all coefficients are appropriately defined via derivatives of the exact field form (see e.g. (7), (11) in [2]). The zeroth-order term balances gravity at (and actually defines the value of) z_0, while $-\gamma_{(1)} = \gamma \equiv M\, \omega_g^2$ is the effective width of the potential well.

Retaining only the linear contribution (i.e. $\alpha = \beta = 0$), and considering phonons of the type: $x_n = A_n\, exp[i\, (knr_0 - \omega t)] + c.c.$, an optical-mode-like dispersion is obtained:

$$\omega^2 = \omega_g^2 - 4\omega_0^2 \sin^2 \frac{kr_0}{2} \tag{3}$$

(cf. (4) in [2]). We will not go into further details concerning the linear regime, since it is covered in the references. Let us now see what happens if the *nonlinear* terms are retained. For simplicity, we shall limit ourselves to the continuum limit, considering wavelengths λ significantly larger than the inter-grain distance r_0 (i.e. $k\, r_0 \ll 1$). With all the above considerations, eq. (1) takes the form:

$$\frac{d^2u}{dt^2} + c_0^2\, \frac{d^2u}{dx^2} + \gamma\, u + \alpha\, u^2 + \beta\, u^3 = 0 \tag{4}$$

where we set $\delta z \equiv u(x,t)$ for simplicity; $c_0 = \omega_0\, r_0$ is a characteristic propagation velocity related to the Debye potential (see (2)); the nonlinear coefficients α, β are related to the electric field: $\alpha = -\gamma_{(2)}/M$, $\beta = -\gamma_{(3)}/M$. Remember that inter-particle interactions are *repulsive*, hence the difference from the nonlinear Klein-Gordon equation used to describe 1d oscillator chains. Phonons in this chain are stable only in the presence of the electric field (i.e. for $\gamma \neq 0$).

We now proceed by considering small-amplitude oscillations of the form:

$$u = \epsilon\, u_1 + \epsilon^2\, {u_2}^2 + \,...$$

at each site. Assuming the existence of multiple scales in time and space, i.e. $X_n = \epsilon^n\, x$, $T_n = \epsilon^n\, t$ (n = 0, 1, 2, ...), we develop the derivatives in (4) in powers of the smallness parameter ϵ and then collect the terms arising in successive orders. The equation thus obtained in each order can be solved and substituted to the subsequent order, and so forth. This reductive perturbation technique is a standard procedure in the study of nonlinear wave propagation (e.g. hydrodynamics, nonlinear optics) often used in the description of localized pulse propagation, prediction of instabilities etc. (see [3]).

The procedure outlined above leads to a solution of the type:

$$u(x,t) = \epsilon\,(A\,e^{i\,(kx-\omega t)} + c.c.) + \epsilon^2\,\alpha\left(-\frac{2\,|A|^2}{\omega_g^2} + \frac{A^2}{3\omega_g^2}\,e^{2i\,(kx-\omega t)} + c.c.\right) + \mathcal{O}(\epsilon^3) \qquad (5)$$

where ω obeys a dispersion law of the form:

$$\omega^2 = \omega_g^2 - c_0^2\,k^2 \qquad (6)$$

(cf. (3) linearized around $k \approx 0$). The slowly-varying amplitude $A = A(X_1 - v_g\,T_1)$ moves at the group velocity $v_g = d\omega/dk = -c_0^2\,k/\omega$; it is found to obey a *Nonlinear Schrödinger Equation* (NLSE) of the form:

$$i\,\frac{dA}{dT} + P\,\frac{d^2A}{dX^2} + Q\,|A|^2\,A = 0 \qquad (7)$$

where the 'slow' variables $\{X,T\}$ are $\{X_1,T_2\}$ respectively. The *dispersion coefficient* P is related to the curvature of the phonon dispersion curve (6):

$$P = \frac{1}{2}\,\frac{d^2\omega}{dk^2} = -\frac{c_0^2\omega_g^2}{2\omega^3} \qquad (8)$$

and the *nonlinearity coefficient* Q is related to electric field nonlinearities:

$$Q = \frac{1}{2\omega}\left(\frac{10\alpha^2}{3\omega_g^2} - 3\,\beta\right) \qquad (9)$$

Notice that $P < 0$, given the parabolic form of $\omega(k)$; however, this is only true close to $k = 0$ (continuum case): in the general (discrete) case (see (3)), P changes sign at a critical value of k (the *zero-dispersion point*) in the first Brillouin zone.

In a generic manner, a modulated wave whose amplitude obeys the NLS equation (7), is be unstable to perturbations if $P\cdot Q > 0$. To see this, one may first check that the NLSE accepts the monochromatic solution (Stokes' wave): $A(X,T) = A_0\,e^{iQ|A_0|^2T} + c.c.$. The standard stability analysis then shows that a linear perturbation of frequency Ω and wavenumber κ will obey:

$$\Omega^2(\kappa) = P^2\,\kappa^2\left(\kappa^2 - 2\frac{Q}{P}\,|A_0|^2\right) \qquad (10)$$

and is therefore expected to grow, for $\kappa \geq \kappa_{cr} = (Q/P)^{1/2}\,|A_0|$, at a rate attaining: $\sigma_{max} = Q\,|A_0|^2$ until the wave collapses. Nevertheless, if $P \cdot Q < 0$, this will never occur. This mechanism is known as the *Benjamin-Feir instability* [4].

The NLSE (7) supports pulse-shaped localized solutions (envelope solitons) of the *bright* ($PQ > 0$) or *dark* ($PQ < 0$) type [5]. The former (*continuum breathers*) are:

$$A = (2D/PQ)^{1/2}\, sech[(2D/PQ)^{1/2}\,(X - v_e\,T)]\, \exp[i\,v_e\,(X - v_c\,T)/2P] + c.c. \qquad (11)$$

where v_e (v_c) is the envelope (carrier) velocity and $D = (v_e^2 - 2\,v_e\,v_c)/(4P^2)$; the latter (*holes*) are physically irrelevant and of no importance here.

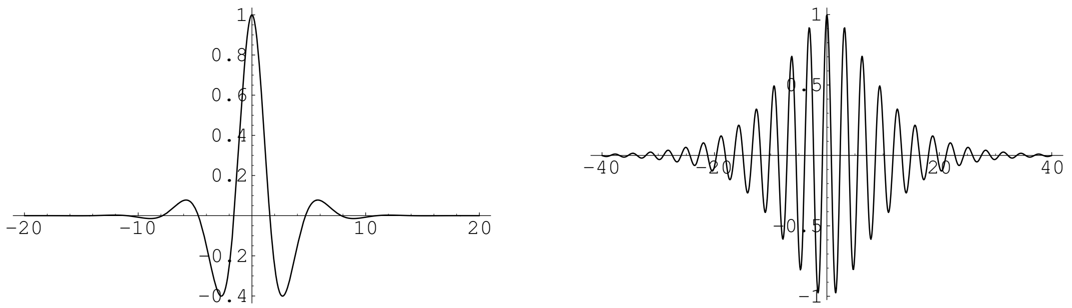

Figure 1: Localized solutions of (7) given by (11) for two different parameter sets.

We see that the occurrence of MI *and* the existence of localized excitations depend on the same criterion, which needs to be thoroughly examined in terms of the electric potential $\phi(z)$. Of course, for a more complete description, one has to take into account transverse to longitudinal mode coupling (ignored in this simple model). Furthermore, in the case of a double-layer crystal, the above picture is strongly modified. A second transverse mode arises [3] and the nonlinear analysis presented here should take into account layer coupling. Work in this direction is in progress and will be reported soon.

References

[1] A number of works in the past have focused on Langmuir-wave propagation related to plasma density perturbations - see e.g. S. Benkadda & V. N. Tsytovich, Phys. Plasmas **2** (3) 2970 (1995) - yet not on 1d DP crystals.

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