Self-Consistent Dynamics of Flute Mode Turbulence with Large-Scale Flows

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It is generally recognized that nonlinear interaction of short-scale fluctuations in a magnetized plasma can generate low-frequency, large-scale nonlinear structures (so called zonal flows and streamers) that play an important role in controlling plasma transport properties in magnetic confinement systems [1-3]. Zonal flows are poloidally and toroidally symmetric structures with a finite radial-scale, significantly larger than the scale of the underlying small-scale turbulence, $q_r << k_r$ (**q** and **k** are the wave vectors for the large-scale perturbations and the background small-scale turbulence). Streamers are convective cells which have short poloidal extent, $q_\theta << k_\theta$, and are radially elongated structures. Zonal flows are believed to act as benign structures that inhibit such transport by decorrelating the radial step length. Streamers, however, are ineffective at inhibiting radial transport and due to their long radial correlation lengths may in fact lead to enhanced and/or bursting levels of transport. On the other hand, these large-scale structures can modulate and regulate the turbulence dynamics themselves via nonlinear coupling to the small-scale fluctuations.

Flute modes are low frequency oscillations of a weakly inhomogeneous magnetized plasma that are uniform in the direction of the external magnetic field $\mathbf{B}_0=(0,0,B_0)$. The curvature of magnetic field lines is imitated here by the fictional gravity $\mathbf{g}=(g,0,0)$. Assuming the electric field to be potential, $\mathbf{E}=-\nabla\Phi$, and expressing the plasma density as $N(\mathbf{r},t)=n_0(x)+\delta n(\mathbf{r},t)$ with $\mathbf{r}\equiv(x,y)$, the two-fluid macroscopic equations are reduced to a pair of coupled nonlinear equations for the dimensionless density $n=\delta n/n_0$ and the potential Φ :

$$\frac{\partial n}{\partial t}+\frac{\kappa c}{B_0}\frac{\partial\Phi}{\partial y}=-\frac{c}{B_0}\{\Phi,n\}, \tag{1}$$

$$\frac{c}{B_0}\left[\frac{\partial}{\partial t}\nabla_\perp^2\Phi-\mathrm{v}_*\frac{\partial}{\partial y}\nabla_\perp^2\Phi\right]+g\frac{\partial n}{\partial y}=-\frac{c^2}{B_0^2}\left[\left\{\Phi,\nabla_\perp^2\Phi\right\}+\frac{T}{e}div\{n,\nabla_\perp\Phi\}\right]. \tag{2}$$

Here $T_i=T_e=T$, $\mathrm{v}_*=\dfrac{\kappa cT}{eB_0}$, $\kappa\equiv-\dfrac{d\ln n_0}{dx}>0$, $g=\dfrac{T}{m_iR}$, and R is the characteristic scale length of the magnetic field inhomogeneity, $\{a,b\}=\mathbf{z}\cdot[\nabla a\times\nabla b]$ with $\mathbf{z}$ denoting the direction of the external magnetic field and $\nabla\equiv\nabla_\perp$. The finite ion Larmor radius (FLR) effect is incorporated to the lowest order by the term proportional to T.

Linearizing Eqs.(1)-(2) for small perturbations $(n,\Phi)\sim\exp(-i\omega t+i\mathbf{k}\cdot\mathbf{r})$, we obtain the dispersion relation for the flute modes with the wave (linear) eigenfrequency

$$\omega_k=\omega_k^{\mathrm{Re}}+i\gamma_k,\quad \omega_k^{\mathrm{Re}}=-\frac{k_y\mathrm{v}_*}{2},\quad \gamma_k=\frac{k_y\mathrm{v}_*}{2}\left(\frac{4L_n}{R}\frac{1}{k^2\rho_i^2}-1\right)^{1/2}, \tag{3}$$

where $k^2 = k_x^2 + k_y^2,\ k >> \kappa$, and $\rho_i = V_{Ti}/\Omega_i$ is the ion Larmor radius. For $T = 0$ Eq.(3) describes pure unstable flute modes with a growth rate $\gamma_0 = \frac{k_y}{k}\sqrt{\kappa g}$. Accounting for the FLR effect leads to the stabilization of flute instabilities for $k^2 \rho_i^2 \geq 4 L_n / R$, where $L_n \equiv \kappa^{-1}$.

To describe the evolution of the coupled system we represent both the electrostatic potential and the plasma density as a sum of a large-scale flow quantity and a small-scale turbulent part, $\Phi = \overline{\Phi} + \widetilde{\Phi}, \quad n = \overline{n} + \widetilde{n}$, and find

$$\frac{\partial \overline{n}}{\partial t} + \frac{\kappa c}{B_0}\frac{\partial \overline{\Phi}}{\partial y} = -\frac{c}{B_0}\overline{\left\{\widetilde{\Phi}, \widetilde{n}\right\}}, \tag{4}$$

$$\frac{c}{B_0}\left[\frac{\partial}{\partial t}\nabla_\perp^2 \overline{\Phi} - \mathrm{v}_* \frac{\partial}{\partial y}\nabla_\perp^2 \overline{\Phi}\right] + g\frac{\partial \overline{n}}{\partial y} = -\overline{R^\Phi} - \overline{R^n}. \tag{5}$$

$$R^\Phi = \frac{c^2}{B_0^2}\left\{\widetilde{\Phi}, \nabla_\perp^2 \widetilde{\Phi}\right\}, \quad R^n = \frac{c^2 T}{e B_0^2}\left\{\widetilde{n}, \nabla_\perp^2 \widetilde{\Phi}\right\} + \frac{c^2 T}{e B_0^2}\left\{\nabla_\perp \widetilde{n}, \nabla_\perp \widetilde{\Phi}\right\}.$$

Thus, a small-scale turbulence can drive, via the standard (R^Φ) and diamagnetic (R^n) Reynolds forces, large-scale flows.

The pattern of the flow, in turn, modulates and regulates the turbulence dynamics:

$$\frac{\partial \widetilde{n}}{\partial t} + \frac{\kappa c}{B_0}\frac{\partial \widetilde{\Phi}}{\partial y} = -\frac{c}{B_0}\left\{\overline{\Phi}, \widetilde{n}\right\} - \frac{c}{B_0}\left\{\widetilde{\Phi}, \overline{n}\right\}, \tag{6}$$

$$\begin{aligned} &\frac{c}{B_0}\left[\frac{\partial}{\partial t}\nabla_\perp^2 \widetilde{\Phi} - \mathrm{v}_* \frac{\partial}{\partial y}\nabla_\perp^2 \widetilde{\Phi}\right] + g\frac{\partial \widetilde{n}}{\partial y} = \\ &-\frac{c^2}{B_0^2}\left[\left\{\overline{\Phi}, \nabla_\perp^2 \widetilde{\Phi}\right\} + \left\{\widetilde{\Phi}, \nabla_\perp^2 \overline{\Phi}\right\}\right] - \frac{c^2 T}{e B_0^2}\left[div\left\{\overline{n}, \nabla_\perp \widetilde{\Phi}\right\} + div\left\{\widetilde{n}, \nabla_\perp \overline{\Phi}\right\}\right] \end{aligned} \tag{7}$$

The system of Eqs. (4)-(7) describes flow structures with $q_z = 0$ and addresses both the zonal flow and streamer formation from a small-scale turbulence on an equal footing as well as the modulation of the flute turbulence by large-scale flows.

It is easy to show that the system of equations has an exact integral

$$\mathsf{E} = \int\left(\left(\overline{n}^2 + \widetilde{n}^2\right) - \rho_i^2 \frac{R}{L_n}\frac{e^2}{T^2}\left((\nabla\overline{\Phi})^2 + (\nabla\widetilde{\Phi})^2\right)\right)dxdy = const \tag{8}$$

representing the total energy conservation. Thus, for the system of flute modes + large-scale structures, small-scales are modulated by larger-scale shear flows so that energy in the small-scale component is not conserved. Neglecting the higher spatial derivatives of the large-scale perturbations, it can be demonstrated that the enstrophy in the system is conserved for the small-scale component alone,

$$\mathsf{I} = \int\left(\left(\nabla\widetilde{n}\right)^2 - \rho_i^2 \frac{R}{L_n}\frac{e^2}{T^2}\left(\Delta\widetilde{\Phi}\right)^2\right)dxdy = const\,. \tag{9}$$

Upon averaging over the fast scales, the evolution equations for zonal flows, Eqs.(6) and (7), become decoupled and are reduced to

$$\frac{\partial \overline{n}}{\partial t} = 0, \tag{10}$$

$$\frac{\partial \overline{\Phi}}{\partial t} = \frac{c}{B_0}\sum_k k_x k_y \left(1 - \frac{R}{2L_n}k^2\rho_i^2\right)\left|\widetilde{\Phi}_k\right|^2. \tag{11}$$

As seen from Eq.(10) the mean plasma density associated with zonal flows does not evolve with time, i.e. it can be considered as a constant.

The evolution equations for streamers remain, however, coupled and have the form

$$\frac{\partial \overline{n}}{\partial t} + iq_y \frac{\kappa c}{B_0}\overline{\Phi} = 0, \tag{12}$$

$$\left(\frac{\partial}{\partial t} - iq_y \mathrm{v}_*\right)\overline{\Phi} - i\frac{gB_0}{q_y c}\overline{n} = -\frac{c}{B_0}\sum_k k_x k_y \left(1 - \frac{R}{2L_n}k^2\rho_i^2\right)\left|\tilde{\Phi}_k\right|^2. \tag{13}$$

The analysis of Eqs. (11) and (13) shows that diamagnetic effects significantly modify the total Reynolds stress. Indeed, a disbalance between standard and diamagnetic Reynolds stresses is required for large-scale flow generation as well. Moreover, the structure of the flow is controlled by the structure of the non-linear coupling to the small-scale fluctuations as well as by the spectral properties of the turbulence and its anisotropy.

The propagation of flute modes in a weakly inhomogeneous medium with slowly varying parameters is conveniently described with the help of a wave kinetic equation for the wave action density. However, the standard expression for the wave action used to describe self-interaction between small-scale fluctuations without the shear flow is modified by the flow and may not be suitable for a system with a mean flow. It is possible to obtain the wave kinetic equation for the flute mode turbulence with slowly varying parameters due to the large-scale flows,

$$\frac{\partial N_k}{\partial t} + \frac{\partial \omega_{k,r}^{NL}}{\partial \mathbf{k}}\frac{\partial N_k}{\partial \mathbf{r}} - \frac{\partial \omega_{k,r}^{NL}}{\partial \mathbf{r}}\frac{\partial N_k}{\partial \mathbf{k}} = S(N_k), \tag{14}$$

corresponding to the conservation of an action-like ("pseudo-action") invariant

$$N_k = k^2\rho_i^2 \frac{4R}{L_n}\left(1 - k^2\rho_i^2 \frac{R}{4L_n}\right)\left|\frac{e\tilde{\Phi}_k}{T}\right|^2. \tag{15}$$

The "collisional" term $S(N_k)$, symbolically, accounts for the wave growth and damping due to linear and nonlinear mechanisms, as well local wave interactions. Below we assume that equilibrium small-scale turbulence is close to a stationary state, so that $S \to 0$. The linear frequency of flute modes entering this equation is modified in the presence of the flows because of the Doppler shift induced by the flow velocity,

$$\omega_{k,r}^{NL} = \omega_k^{\mathrm{Re}} + \mathbf{k}(\mathbf{V}_0 + \mathbf{V}_1), \quad \mathbf{V}_0 = -\frac{c}{2B_0}\left[\nabla\overline{\Phi}\times\mathbf{z}\right], \quad \mathbf{V}_1 = -\frac{cT}{2eB_0}\left[\nabla\overline{n}\times\mathbf{z}\right]. \tag{16}$$

Here $\mathbf{V}_0$ is associated with the Doppler effect introduced by $\boldsymbol{E}\times\boldsymbol{B}$ flows and $\mathbf{V}_1$ corresponds to the lowest-order FLR corrections due to finite ion temperature. Thus, the small-scale turbulence will be sheared by both the flow shear and the diamagnetic effect.

Eq.(14) generalizes the wave kinetic equation for the case of the unstable flute modes in the presence of the mean plasma flow. A coupled system of Eqs.(4),(5) and (14) can be treated as "predator-prey" system that self-consistently describes these two disparate components of wave turbulence: the population of waves (prey) growing via linear instability, generates large-scale structures (predator) through the Reynolds stress. Concomitantly, the large-scale structures (predator) regulate the wave population (prey).

The instability of large-scale flows can be obtained by linearizing Eqs.(4),(5) and (14) for small perturbations. We assume that the wave spectrum consists of an equilibrium part N_k^0, subject to stability analysis, and a perturbed part $\tilde{N}_k$, i.e. $N_k = N_k^0 + \tilde{N}_k$. The quantity $\tilde{N}_k$ may be straightforwardly calculated via linearization of the wave kinetic equation (14):

$$\frac{\partial \widetilde{N}_k}{\partial t} + \mathbf{V}_g \frac{\partial \widetilde{N}_k}{\partial \mathbf{r}} - \frac{\partial}{\partial \mathbf{r}} \left(\mathbf{k} \left(\mathbf{V}_0 + \mathbf{V}_1 \right) \right) \frac{\partial N_k^0}{\partial \mathbf{k}} = 0 . \tag{17}$$

Assuming the perturbations to be of the form $\left(\widetilde{N}_k, \overline{\Phi}, \overline{n}\right) \sim \exp\left(-i\Omega T + i\mathbf{q}\cdot\mathbf{r}\right)$, and using Eq.(15), the growth rate of zonal flows ($q_x \neq 0$) is calculated as

$$\gamma^{ZF} = -\frac{c^2 T^2}{8e^2 B_0^2} \int d\mathbf{k} \frac{k_x k_y^2 q_x^4}{k^2 \rho_i^2} \frac{\partial N_k^0}{\partial k_x} R(\Omega, q). \tag{18}$$

In contrast to zonal flows, streamer structures ($q_y \neq 0$) have a real frequency,

$$\Omega^S = -\frac{q_y \mathrm{v}_*}{2} \left(1 \mp \sqrt{1 - \frac{4L_n}{R} \frac{1}{q_y^2 \mathrm{v}_*^2}} \right), \tag{19}$$

and their growth rate is

$$\gamma^S = -\frac{\Omega^S}{2\Omega^S + q_y \mathrm{v}_*} \frac{c^2 T^2}{8e^2 B_0^2} \int d\mathbf{k} \frac{k_x^2 k_y q_y^4}{k^2 \rho_i^2} \frac{\partial N_k^0}{\partial k_y} R(\Omega, q). \tag{20}$$

Here $R(\Omega,q) = i/\left(\Omega - \mathbf{q}\mathbf{V}_g + i\Delta\omega_k\right)$ is the response function with $\mathbf{V}_g$ denoting the flute mode group velocity and $\Delta\omega_k$ the total decorrelation frequency which may also include the linear growth rate and a nonlinear frequency shift. While $R(\Omega,q) \to \pi\delta\left(\Omega - \mathbf{q}\mathbf{V}_g\right)$ in the weakly nonlinear regime, a wide spectrum of fluctuations is associated with $R(\Omega,q) \to 1/\Delta\omega_k$. Note that the conditions for growth, $\gamma > 0$, are equivalent to

$$k_x \frac{\partial N_k^0}{\partial k_x} < 0 \quad \text{and} \quad k_y \frac{\partial N_k^0}{\partial k_y} < 0 , \tag{21}$$

for zonal flows and streamers, correspondingly. These conditions are typically satisfied in flute mode turbulence, requiring therefore no population inversion. Hence this instability is a manifestation of an inverse cascade and shows that energy is pumped into longer scales.

Our study, in which we presented the self-consistent theory of large-scale structure generation by flute mode turbulence, shows that a small-scale turbulence can drive large-scale flow via Reynolds stresses, provided that a disbalance between standard and diamagnetic Reynolds stresses exists. Diamagnetic effects can significantly modify the total Reynolds force, and can also suppress the generation of large-scale structures. The pattern of the flow, in turn, modulates and regulates the turbulence dynamics. The flow structure is controlled by the structure of non-linear coupling to the small-scale fluctuations as well as by the spectral properties of the turbulence and its anisotropy. Wave kinetic equation is formulated and a structure of an appropriate adiabatic invariant for small-scale turbulence in the presence of a mean flow is determined. The rates at which large-scale structures grow via resonant type instability are calculated for both zonal flows and streamer structures.

Acknowledgements

This work was supported by the European Communities under an association contract between EURATOM and the Swedish Research Council (VR), Grant No F 5102-8092001. Further it has been partially carried out within the Association EURATOM-OEAW project P4. The content of the publication is the sole responsibility of its authors and does not necessarily represent the views of the European Commission or its services.

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