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Nonlinear Covariant Gyrokinetic Equations

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A closed set of relativistic gyrokinetic equations, consisting of the collisionless gyrokinetic equation and the phase-independent expressions for charge and current densities, is derived for an arbitrary four-dimensional coordinate system. The guiding-center dynamics of charged particles and the gyrokinetic transformation are obtained accurate through second order of the ratio of the Larmor radius to the gradient length. The wave-terms ($k\rho \sim 1$) are described in the second-order approximation with respect to the amplitude of the wave. The same approximations are used in the derivation of the gyrophase-averaged charge and current densities. Averaging is explicit.

Covariant formulation allows the derived equations to be easily rendered for any coordinate system in four-dimensional Riemann space-time. It is important for astrophysics applications (the gravitational field is included self-consistenly,) as well as for problems where description in curvilinear magnetic coordinates is convenient. The covariant formulation of the theory, i.e. with relation to *any* reference frame, is inherently more general and symmetric than the non-relativistic treatment and its "relativistic" generalizations. As a result, even the non-relativistic limit of the theory is found to have somewhat broader applicability range than the standard derivation.

In our previous paper [1] the development of the covariant theory has been carried out through first order in the expansion parameter, and without the wave fields. Covariant theory by Boghosian [2] is derived by sequential Lee transformations, lacks nonlinear terms, and has restrictions on the electric field. Our derivation is based on the perturbative Lagrangian approach with a fully relativistic, four-dimensional covariant formulation. Its results are algebraically simple and improve on existing limitations of the current gyrokinetic theory (due to internal symmetry of the electromagnetic field in four-dimensional formulation.)

Approach

The motion of a particle with the rest-mass m_a and charge q_a in prescribed fields in phase space can be found from the Hamilton variational principle $\delta S = 0$, as the extremal of the functional[1]

$$S = \int Q_\mu \mathrm{d}x^\mu = \int (qA_\mu(x^\nu) + u_\mu)\mathrm{d}x^\mu, \tag{1}$$

where $q = q_a/m_a c^2$, and variations of u_μ occur on the hypersurface $u_\mu u^\mu = 1$.

Assume that the gradient lengths are much larger than the Larmor radius. Allow for existence of wave-fields with sharp gradients [$k\rho \sim O(1)$, including $k_\parallel \rho \sim O(1)$,] and rapidly varying in time [$\omega\rho/c \sim O(1)$], but small amplitude, according to the ordering scheme[3]:

$$Q_\mu \mathrm{d}x^\mu = \{u_\mu + q(\frac{1}{\varepsilon}A_\mu + \lambda a_\mu)\}\mathrm{d}x^\mu, \tag{2}$$

where ε and λ are formal small parameters allowing distinction between the large-scale background field A_μ, and the wave-fields given by a_μ. We search for the gyrokinetic transformation $(y^i) \equiv (x'^\alpha, \phi, \widehat{\mu}, u_\parallel) \leftrightarrow (x^\alpha, u^\beta)$ as

$$x^\nu = x'^\nu + \sum_{s=1} \varepsilon^s r_s^\nu(y^i), \tag{3}$$

where ϕ is the gyrophase, x'^ν is the 4-vector "guiding center" position, $\mathbf{r}_s$ are arbitrary 4-vector functions of the new variables (y^i) to be determined. We require that $\mathbf{r}_s$ are purely oscillatory in ϕ, i.e. the ϕ-averages of $\mathbf{r}_s$ are zero, as a part of the x'^ν- definition.

To define the rest of the gyrokinetic transformation, we first introduce an orthogonal basis of unit 4-vectors $(\tau, \mathbf{l}, \mathbf{l}', \mathbf{l}'')$ so that the last three 4-vectors are space-like. A special choice of orientation links the basis $(\tau, \mathbf{l}, \mathbf{l}', \mathbf{l}'')$ to the electromagnetic field tensor, $F_{\mu\nu} = \partial A_\nu/\partial x^\mu - \partial A_\mu/\partial x^\nu$. With this choice the $(\mathbf{l}', \mathbf{l}'')$-plane coincides with the space-like invariant plane of the antisymmetric tensor $F_{\mu\nu}$. Then if $(\mathbf{l}', \mathbf{l}'')$ is the first invariant plane of $F_{\mu\nu}$, then $(\mathbf{l}, \tau)$ is the other, and if H and E are the eigenvalues of $F_{\mu\nu}$, then

$$F_{\mu\nu} l''^\nu = H l'_\mu, \quad F_{\mu\nu} l'^\nu = -H l''_\mu, \quad F_{\mu\nu} l^\nu = E\tau_\mu, \quad F_{\mu\nu}\tau^\nu = E l_\mu. \tag{4}$$

The four-velocity in the new variables is defined as

$$u_\mu = w\left(l'_\mu \cos\phi + l''_\mu \sin\phi\right) + \bar{u}_\mu, \tag{5}$$

which can be regarded as the definition for the gyrophase ϕ: it is defined as *an angle in the velocity-subspace,* where we introduce the cylindrical coordinate system. This definition is covariant. The ϕ-independent part of the 4-velocity $\bar{\mathbf{u}}$ is not completely arbitrary, but satisfies certain restrictions following from $u_\mu u^\mu = 1$ for all ϕ:

$$\bar{u}_\mu = u_\parallel l_\mu + u_o \tau_\mu, \qquad u_o^2 = 1 + w^2 + {u_\parallel}^2 \tag{6}$$

. Any two of three scalar functions w, u_o or $u_\parallel$ can be considered independent characteristics of velocity, while the third can be expressed via (6).

Evaluating the Lagrangian in new variables and requiring it to be independent of ϕ, we arrive at the form of the gyrokinetic transformation and the new Lagrangian.

Results

The transformed variational principle is found in the second order in λ and second order in ε, i.e. with terms of the order $\varepsilon^2\lambda^2$ and ε^2 retained: $\delta S = 0$ yields the particle phase-space trajectory with

$$S = \int \left(qA_\mu(\mathbf{x}') + u_\parallel l_\mu + \left(1 + 2qH^*\hat{\mu} + {u_\parallel}^2\right)^{1/2} \tau_\mu + q\overline{a_\mu} + \frac{1}{2}\hat{\mu}\chi_\mu\right) \mathrm{d}x'^\mu + \hat{\mu}\mathrm{d}\phi, \tag{7}$$

where $(x'^\mu, u_\parallel, \hat{\mu}, \phi)$ or $(x'^\mu, u_\parallel, w, \phi)$ are the new gyrokinetic variables with

$$\hat{\mu} = w^2/2qH^* + \hat{\mu}^{(2)},$$

$$\overline{a_\mu} = \frac{1}{(2\pi)^2} \int \mathrm{d}^4k\ a_\mu(\mathbf{k}) e^{i\mathbf{kx}'} J_0(\xi) + \overline{a_\mu}^{(1)}, \tag{8}$$

is the averaged wave-field potential,

$$H^* = H\left(1 + \frac{1}{2\pi^2}\int \mathrm{d}^4k\ \frac{f_{\mu\nu}(\mathbf{k})l'^{\nu}l''^{\mu}}{H}\frac{J_1(\xi)}{\xi}e^{i\mathbf{k}\mathbf{x}'}\right), \tag{9}$$

$$\xi = k_\perp\rho = \left(2\hat{\mu}\left[(k_\nu l'^\nu)^2 + (k_\nu l''^\nu)^2\right]/qH\right)^{1/2};\ f_{\mu\nu} = \partial a_\nu/\partial x^\mu - \partial a_\mu/\partial x^\nu.$$

$$\chi_\mu = l'_\nu\frac{\partial l''^\nu}{\partial x'^\mu} - l''_\nu\frac{\partial l'^\nu}{\partial x'^\mu} - (l'^\nu l'^\varsigma + l''^\nu l''^\varsigma)\frac{1}{H}\frac{\partial F_{\mu\nu}}{\partial x'^\varsigma} \tag{10}$$

describes the inhomogeneity of the electromagnetic field.

The second-order (nonlinear) corrections look like

$$\overline{a_\mu}^{(1)} = \frac{i}{(2\pi)^4}\int\int \mathrm{d}^4k\mathrm{d}^4k'\ e^{i(\mathbf{k}+\mathbf{k}')\mathbf{x}'}a_\mu(\mathbf{k})k_\nu D^{\nu\eta}a_\eta(\mathbf{k}')\left[J_0(\xi'') - J_0(\xi)J_0(\xi')\right]. \tag{11}$$

where $D^{\nu\mu}$ is the inverse of $F_{\mu\nu}$, $\xi' = \xi(\mathbf{k}')$; $\xi'' = \xi(\mathbf{k}+\mathbf{k}')$;

$$\hat{\mu}^{(2)} = \frac{w^2}{qH^2}\frac{1}{(2\pi)^4}\int\int \mathrm{d}^4k'\mathrm{d}^4k\ a_\mu(\mathbf{k})a_\nu(\mathbf{k}')e^{i(\mathbf{k}+\mathbf{k}')\mathbf{x}'}R^{\mu\nu},$$

where

$$R^{\mu\nu} = \left(l'^\mu l''^\zeta - l''^\mu l'^\zeta\right)\left[\frac{J_1(\xi'')}{\xi''}\left(k_\zeta + k'_\zeta\right) - J_0(\xi')\frac{J_1(\xi)}{\xi}k_\zeta\right]k_\eta D^{\eta\nu} + $$
$$+\frac{J_1(\xi'')}{2\xi''}k_\eta k'_\zeta(l'^\eta l''^\zeta - l'^\zeta l''^\eta)D^{\nu\mu}. \tag{12}$$

The four equations of motion can be cast in the form

$$\left(qHl''_\mu + l'^\nu T_\nu\left[u_\parallel l_\mu + u_0\tau_\mu + q\overline{a_\mu} + \frac{1}{2}\hat{\mu}\chi_\mu\right]\right)\mathrm{d}x'^\mu = 0, \tag{13}$$

$$\left(-qHl'_\mu + l''^\nu T_\nu\left[u_\parallel l_\mu + u_0\tau_\mu + q\overline{a_\mu} + \frac{1}{2}\hat{\mu}\chi_\mu\right]\right)\mathrm{d}x'^\mu = 0, \tag{14}$$

$$\mathrm{d}u_\parallel - qE\tau_\mu\mathrm{d}x'^\mu + l^\nu T_\nu\left[u_\parallel l_\mu + u_0\tau_\mu + q\overline{a_\mu} + \frac{1}{2}\hat{\mu}\chi_\mu\right]\mathrm{d}x'^\mu = 0. \tag{15}$$

$$\left(u_0 l_\mu + u_\parallel\tau_\mu\right)\mathrm{d}x'^\mu = 0. \tag{16}$$

The first two equations describe the drift motion, where the operator T_ν is defined by $T_\nu[y_\mu] \equiv \partial y_\mu/\partial x'^\nu - \partial y_\nu/\partial x'^\mu$. The last two equations determine the parallel velocity and the energy conservation.

The collisionless kinetic equation can be represented in the parametrization - independent form as

$$\frac{\partial f}{\partial x^\mu}\mathrm{d}x^\mu + \frac{\partial f}{\partial u_\nu}\mathrm{d}u_\nu = 0,$$

where the differentials are tangent to the particle orbit. In the usual way it can be transformed into the gyrokinetic equation

$$\frac{\partial f}{\partial x'^{\mu}}\mathrm{d}x'^{\mu}+\frac{\partial f}{\partial u_{\parallel}}\mathrm{d}u_{\parallel}=0, \tag{17}$$

where the differentials are defined by equations (13)-(16).

The electromagnetic field is governed by Maxwell's equations with self-consistently-defined 4-current density j^{μ}

$$\frac{\partial}{\partial x^{\nu}}\left(\sqrt{-g}F^{\mu\nu}\right)=-\frac{4\pi}{c}\sqrt{-g}j^{\mu}=Q^{\mu}(\mathbf{x}), \tag{18}$$

which can be expressed via the gyrokinetic distribution function as

$$Q^{\mu}(\mathbf{x})=-4\pi\sum_{\alpha}q_{\alpha}\int\left[w\left(l'^{\mu}\cos\phi+l''^{\mu}\sin\phi\right)+u_{\parallel}l^{\mu}+u_{o}\tau^{\mu}\right]f_{\alpha}(\mathbf{x}-\sum_{i=1}\varepsilon^{i}\mathbf{r}_{i})\frac{w\mathrm{d}w\mathrm{d}\phi\mathrm{d}u_{\parallel}}{u_{o}}.$$

It can be evaluated by orders of ε,λ as

$$Q^{\mu}_{(00)}=-8\pi^{2}\sum_{\alpha}\frac{q_{\alpha}^{2}}{m_{\alpha}c^{2}}H^{*}(x^{\mu})\int\left(\frac{u_{\parallel}}{u_{o}}l^{\mu}+\tau^{\mu}\right)f_{\alpha}^{(0)}\left(x^{\mu},\widehat{\mu},u_{\parallel}\right)\mathrm{d}\widehat{\mu}\mathrm{d}u_{\parallel}, \tag{19}$$

$$Q^{\mu}_{(10)}=-8\pi^{2}\sum_{\alpha}\frac{q_{\alpha}^{2}}{m_{\alpha}c^{2}}\int\frac{\widehat{\mu}}{\sqrt{H}}\left(l'^{\nu}\frac{\partial}{\partial x^{\nu}}\left(\frac{H^{3/2}f_{\alpha}^{(0)}}{u_{o}}l''^{\mu}\right)-l''^{\nu}\frac{\partial}{\partial x^{\nu}}\left(\frac{H^{3/2}f_{\alpha}^{(0)}}{u_{o}}l'^{\mu}\right)\right)\mathrm{d}\widehat{\mu}\mathrm{d}u_{\parallel}, \tag{20}$$

$$Q^{\mu}_{(01)}=-2\sum_{\alpha}\frac{q_{\alpha}^{2}}{m_{\alpha}c^{2}}H\int\mathrm{d}\widehat{\mu}\mathrm{d}u_{\parallel}\int\mathrm{d}^{4}kf_{\alpha}^{(1)}\left(\mathbf{k},\widehat{\mu},u_{\parallel}\right)e^{i\mathbf{kx}}\left\{\left(\frac{u_{\parallel}}{u_{o}}l^{\mu}+\tau^{\mu}\right)J_{0}\left(\xi\right)+\right.$$
$$\left.+\frac{2\widehat{\mu}H}{u_{o}}(l''^{\mu}l'^{\nu}-l''^{\nu}l'^{\mu})i\varepsilon k_{\nu}\frac{J_{1}\left(\xi\right)}{\xi}\right\}, \tag{21}$$

etc. Here $f_{\alpha}^{(1)}$ is the Fourier-decomposed part of the distribution function caused by the wave.

References

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