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THE NONLINEAR SMALL-SCALE DYNAMO AND ISOTROPIC MHD TURBULENCE

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Homogeneous incompressible MHD turbulence has been studied in the literature in two main regimes: with and without an externally imposed uniform mean magnetic field. In the former, explicitly anisotropic, case, Kolmogorov-style phenomenologies [3, 2] predict a state of detailed scale-by-scale equipartion between hydrodynamic and magnetic fluctuations, which are Alfvén waves propagating along the mean field. With proposed scalings for the resulting spectra ranging between k^{-2} and $k^{-5/3}$, both the hydrodynamic and the magnetic energies are concentrated at the outer (forcing) scale of the turbulence. It has long been the commonly held view that the other, *isotropic*, regime of MHD turbulence would, in the fully developed state, be very similar, with the energetic magnetic fluctuations at the outer scale providing the effective mean field to support the Alfvén waves in the inertial range [3]. The isotropic regime is closely related to the problem of *turbulent MHD dynamo,* where a weak initial magnetic field is embedded into a turbulent conducting medium and its evolution is studied. In view of the arguments above, the dynamo has been expected to culminate in an Alfvénic equipartition state. In what follows, we will see this outcome is less obvious than it seems.

Because of the close correlation between velocities and magnetic fields in the Alfvén regime, the ratio between the fluid viscosity and the magnetic diffusivity (the magnetic Prandtl number $\mathrm{Pr} = \nu/\eta$) has not been considered important, as long as both ν and η were small enough for the turbulence to develop. However, it is a prominent property of hot low-density turbulent astrophysical plasmas (interstellar medium, some accretion discs and jets, protogalaxies, galaxy clusters, early Universe etc.) that their Pr tends to be extremely large and, therefore, that they possess a broad range of subviscous scales at which magnetic fields can exist, while velocities cannot. The magnetic field lines are nearly perfectly frozen into such a highly conducting fluid. The fluid motions, even though restricted to scales above the viscous scale ℓ_ν, can excite magnetic fluctuations at much smaller scales via stretching and folding of the field lines. In the kinematic (weak-field) regime, the result is an exponentially fast pile-up of magnetic energy at the resistive scale $\ell_\eta \sim \mathrm{Pr}^{-1/2}\ell_\nu$, with a $k^{3/2}$ spectrum extending through the subviscous range. The associated time scale is the turnover time of the viscous-scale turbulent eddies [1, 4].

The salient feature of the kinematically generated small-scale fields is their *folding structure:* the fields are "folded" in such a way that the smallness of their characteristic scale is due to rapid transverse spatial oscillation of the field direction, while the field lines remain largely unbent up to the scale of the stretching eddy [6, 5, 8, 9]. Quantitatively, the field structure can be studied in terms of statistics of the field-line curvature $K = |\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}|$ (where $\hat{\mathbf{b}} = \mathbf{B}/B$) and of its correlation with the field strength B. In the kinematic case, analytic theory is possible subject to certain modelling assumptions [8]. The bulk of the PDF of K turns out to be at the velocity scales, with a power tail $\sim K^{-13/7}$ extending through the subviscous range $K \gg \ell_\nu^{-1}$ (this scaling is very well confirmed numerically [8]). This reflects the predominant straightness of the folded field lines at subviscous scales. They are significantly curved only in the "corners" of the folds — the power tail of the curvature PDF describes the intermittent distribution of these corners.

Furthermore, the field strength is anticorrelated with the field-line curvature, i.e., the field is stronger in the straight parts of the folds than in the corners [8, 9].

The most important implication of these results is that the system is in a *reduced-tension state:* the tension force, which controls the nonlinear back reaction, is $\mathbf{B}\cdot\nabla\mathbf{B} \sim KB^2 \sim B^2/\ell_\nu$ during the kinematic stage [8], so the nonlinearity becomes important ($\mathbf{B}\cdot\nabla\mathbf{B} \sim \mathbf{u}\cdot\nabla\mathbf{u}$) when the magnetic energy approaches that of the viscous-scale eddies (in a chaotically tangled field, this would happen at much smaller magnetic energies, because we would have $\mathbf{B}\cdot\nabla\mathbf{B} \sim B^2/\ell_\eta$). Even after the onset of the nonlinear regime, the field in the corners remains weak and cannot be expected to exert a significant amount of back reaction on the flow. In fact, the folding structure, once set up, is generally very hard to undo, as the detailed "unwinding flows" required for that cannot exist at subviscous scales. These arguments are corroborated by the numerical simulations [9], which show the persistence of the folding structure in the nonlinear regime. In particular, both the anticorrelation between the field strength and curvature and the $K^{-13/7}$ subviscous-range scaling of the curvature PDF remain unchanged. The rigidity of the folding structure plays a key role in the further developments.

We now propose the following scenario of the nonlinear-dynamo evolution [7]. The MHD induction equation gives the evolution law for the magnetic energy:

$$\partial_t\langle B^2\rangle = 2\gamma(t)\langle B^2\rangle - 2\eta k_{\rm rms}^2(t)\langle B^2\rangle, \quad \text{where} \quad \gamma(t) = \langle \mathbf{BB} : \nabla\mathbf{u}\rangle/\langle B^2\rangle, \tag{1}$$

and $k_{\rm rms}(t)$ is the spectrum-integrated rms wave number of the magnetic field. In Eq. (1), $\gamma(t)$ can be interpreted as the effective stretching rate at time t. During the kinematic stage, $\gamma(t) \sim u_{\ell_\nu}/\ell_\nu$, the turnover rate of the viscous-scale eddies. Once the magnetic energy becomes comparable to the energy of these eddies, the Lorentz back reaction must act to suppress the stretching motions associated with them. However, the next-larger-scale eddies still have energies above that of the magnetic field, though the turnover rate of these eddies is smaller. These eddies will continue to amplify the field at this slower rate. The scale of these eddies is $\ell_{\rm s} > \ell_\nu$, so the folds are elongated accordingly and the tension force is $\mathbf{B}\cdot\nabla\mathbf{B} \sim B^2/\ell_{\rm s}$. The corresponding inertial term is $\mathbf{u}\cdot\nabla\mathbf{u} \sim u_{\ell_{\rm s}}^2/\ell_{\rm s}$, so these eddies become suppressed when $B^2 \sim u_{\ell_{\rm s}}^2$, whereupon it will be the turn of the next-larger eddies to provide the dominant stretching action. Thus, at any given time, the "stretching scale" $\ell_{\rm s}(t)$ is defined by $u_{\ell_{\rm s}(t)}^2 \sim \langle B^2\rangle(t)$ and $\gamma(t)$ is the turnover rate of the eddies of scale $\ell_{\rm s}(t)$: $\gamma(t) \sim u_{\ell_{\rm s}(t)}/\ell_{\rm s}(t)$. The first term on the rhs of Eq. (1) is then $\gamma(t)\langle B^2\rangle(t) \sim u_{\ell_{\rm s}(t)}^3/\ell_{\rm s}(t) \sim \epsilon$, where $\epsilon = \text{const}$ is the Kolmogorov energy flux. The physical meaning of this result is as follows. The turbulent energy injected at the forcing scale cascades hydrodynamically down to the scale $\ell_{\rm s}(t)$ where (a finite fraction of) it is diverted into the small-scale magnetic fields. We conclude that the magnetic energy should grow linearly with time during this stage, $\langle B^2\rangle(t) \sim \epsilon t$, and $\gamma(t) \sim 1/t$. We stress that the magnetic field is still organized in folds of characteristic length $\ell_{\rm s}(t)$ with direction reversals at the resistive scale ℓ_η. Comparing the two terms in the rhs of Eq. (1), we can estimate $\ell_\eta \sim k_{\rm rms}^{-1}(t) \sim [\gamma(t)/\eta]^{-1/2} \sim (\eta t)^{1/2}$, so ℓ_η increases. Indeed, as the stretching slows down, *selective decay* eliminates the modes at the extreme UV end of the magnetic spectrum for which the resistive time is now shorter than the stretching time and which, consequently, cannot be sustained anymore.

Some fluid motions do survive at scales below $\ell_{\rm s}(t)$ and down to the viscous cutoff. They are Alfvén waves that propagate along the folds of direction-reversing magnetic fields and do not amplify the magnetic energy [7]. Their dispersion relation is $\omega =$

$\pm k_{\|}\sqrt{\langle B^2\rangle}$, where $k_{\|} = \left(\mathbf{kk} : \hat{\mathbf{b}}\hat{\mathbf{b}}\right)^{1/2}$. A finite fraction of the hydrodynamic energy arriving from the large scales is channelled into the turbulence of these waves. Since $\ell_{\rm s}^{-1} \ll k_{\|} \ll \ell_\nu^{-1}$, ω is larger than the resistive-dissipation rate of the small-scale fields: $\omega \gg u_{\ell_{\rm s}}/\ell_{\rm s} \sim \gamma \sim \eta k_{\rm rms}^2$. Therefore, the Alfvén waves are mostly dissipated *viscously* via the MHD turbulent cascade [3, 2], rather than resistively. This enables us to consider the evolution of the small-scale fields separately from that of the Alfvén waves.

The nonlinear-growth stage continues until the magnetic energy becomes comparable to the energy of the outer-scale eddies, $\langle B^2\rangle \sim u_{\ell_0}$, and $\ell_{\rm s} \sim \ell_0 \sim {\rm Re}^{3/4}\ell_\nu$, the forcing scale. Thus, energy equipartition between magnetic and velocity fields is achieved. The time scale for this to happen is the turnover time of the outer-scale eddies $t_0 \sim (u_{\ell_0}/\ell_0)^{-1} \sim {\rm Re}^{1/2}(u_{\ell_\nu}/\ell_\nu)^{-1}$. At this point, $\gamma \sim u_{\ell_0}/\ell_0$. Therefore, the resistive scale is now $\ell_\eta \sim ({\rm Re\,Pr})^{-1/2}\,\ell_0$, which is larger than its kinematic value $\sim {\rm Pr}^{-1/2}\ell_\nu$ by a factor of ${\rm Re}^{1/4}$. As there are no scales in the system larger than ℓ_0, there can be no further growth of the magnetic energy. Stretching and bending the ever more rigid magnetic field becomes increasingly harder and, instead of amplifying the magnetic energy, causes the field to spring back. Therefore, the rate of the energy transfer into the small-scale magnetic field decreases, as this channel of energy dissipation becomes inefficient. Instead, we expect the energy injected by the forcing to be increasingly diverted into the Alfvénic turbulence that is left throughout the inertial range in the wake of the suppression of the stretching motions. During this last stage of the nonlinear dynamo, the magnetic energy stays approximately constant (growing only very slightly) while $\gamma(t)$ drops below the turnover rate of the outer-scale eddies. The energy balance Eq. (1) then implies further movement of $\ell_\eta \sim k_{\rm rms}^{-1}(t)$ towards larger scales. The mechanism for this decrease is the same as in the nonlinear-growth stage: the resistive decay of the high-k modes outpaces their regeneration by the weakened stretching.

Thus, ℓ_η increases both in the nonlinear-growth stage and during the subsequent slower approach to saturation. In [7], we show that the evolution of the magnetic-energy spectrum in both regimes is likely to be *self-similar* with $k_{\rm rms}(t) \sim (\eta t)^{-1/2}$. It is very important to understand how far it can proceed. In our arguments so far, we have disregarded the Alfvénic component of the turbulence. It is not, however, justified to do so once the small-scale magnetic energy reaches the velocity scales. Indeed, the decrease of $k_{\rm rms}$ is basically a consequence of the balance between the field-amplification and the resistive-dissipation terms in Eq. (1). However, once $k_{\rm rms} \sim \ell_\nu^{-1}$, the Alfvénic turbulence will start to affect $k_{\rm rms}$ in an essential way: since the waves are damped at the viscous scale, $k_{\rm rms}$ must stabilize at $k_{\rm rms} \sim \ell_\nu^{-1}$. The resulting turbulent state features folded magnetic fields reversing directions at the viscous scale plus Alfvén waves in the inertial range propagating along the folds. There are then two possibilities: either (i) this represents the final steady state of the isotropic MHD turbulence, or (ii) further evolution will lead to unwinding of the folds and continued energy transfer to larger scales, so that the spectrum will eventually peak at the outer scale and have an Alfvénic power tail extending through the inertial range. It is in the latter case that the turbulence in the inertial range would be of the usual Alfvén-wave kind [3, 2], where the large-scale magnetic fluctuations would provide a mean field along which the inertial-range Alfvén waves propagate. However, in order to achieve such a state, the folds would have to be unwound. In view of the rigidity of the folding structure, it is unclear how this can be done. This dichotomy remains unresolved and requires further study.

In either case, $k_{\rm rms} \sim \ell_\nu^{-1}$, so the fully developed isotropic MHD turbulence is characterized by the equalization of the resistive and viscous scales. The time scale at which such an equalization is brought about is the resistive time associated with the viscous scale of the turbulence: $t_\eta(\ell_\nu) \sim \ell_\nu^2/\eta$. We immediately notice that, in order for the nonlinear-growth stage to run its full course up to the energy equipartion $\langle B^2 \rangle \sim u_{\ell_0}^2$, this time scale must be longer than the turnover time of the outer-scale eddies: $t_\eta(\ell_\nu) \gg t_0$, which requires $\mathrm{Pr} \gg \mathrm{Re}^{1/2}$ and is also equivalent to the condition that $\ell_\eta \ll \ell_\nu$ at the end of the nonlinear-growth stage. This constitutes the "true large-Pr regime," which is the one relevant for astrophysical plasmas. In this regime, the magnetic energy saturates at the equipartition level, $\langle B^2 \rangle \sim u_{\ell_0}^2$. From Eq. (1), we get an estimate of the amount of turbulent power that is still dissipated resisively: $\gamma\langle B^2 \rangle \sim (\eta/\ell_\nu^2)u_{\ell_0}^2 \sim \epsilon\,\mathrm{Re}^{1/2}/\mathrm{Pr} \ll \epsilon$, i.e., most of the injected power now goes into the viscously-dissipated Alfvénic motions. The other possibility is $\mathrm{Pr} \lesssim \mathrm{Re}^{1/2}$. In this case, the self-similar dynamo evolution is curtailed during the nonlinear-growth stage with magnetic energy still at a subequipartition value, $\langle B^2 \rangle/u_{\ell_0}^2 \sim t_\eta(\ell_\nu)/t_0 \sim \mathrm{Pr}/\mathrm{Re}^{1/2}$, and resistivity continuing to take a significant part in the dissipation of the turbulent energy. If no further evolution takes place, this gives an estimate of the saturation energy of the magnetic component of the turbulence. In this regime, Pr is not large enough to capture all of the physics of the large-Pr dynamo. Most of the extant numerical simulations appear to be in this regime (see references in [7]).

Thus, we have identified two possibilities for the long-time behaviour of the isotropic MHD turbulence: saturation in the usual Alfvénic state [3, 2] and saturation with the magnetic energy tied up in the viscous-scale fields. Which one is realized depends on the way the small-scale folded fields interact with the inertial-range Alfvénic turbulence. We stress that there is no numerical evidence available at present that would confirm that, the isotropic MHD turbulence *without externally imposed mean field* — at any Prandtl number — attains the Alfvénic state of scale-by-scale equipartition envisioned in the commonly accepted phenomenologies [3, 2]. In fact, medium-resolution simulations [5] rather seem to support the final states with small-scale energy concentration even for $\mathrm{Pr} = 1$. This does not mean that the Alfvénic picture is incorrect *per se*. However, all existing phenomenologies of the Alfvénic turbulence depend on the assumption [3] that the strongest magnetic fields are those at the outer scale. This is authomatically satisfied if a finite mean field is imposed externally. However, it remains to be seen if such a distribution of energy is set up self-consistently when the turbulence is isotropic.

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