29th EPS Conference on Plasma Phys. and Contr. Fusion Montreux, 17-21 June 2002 ECA Vol. ***26B****, P-4.033 (2002)*

To the Theory of the Ballooning Perturbations in the Inner Magnetosphere of the Earth

Cheremnykh O.K., Burdo O.S., Parnowski A.S.

Space Research Institute NASU & NSAU, Kyiv, Ukraine

The low-frequency MHD waves in the inner magnetosphere of the Earth have been studied using the set of equations of small oscillations obtained from the ideal MHD equations and the dipole model of the geomagnetic field. This set describes both low-scale and large-scale disturbances in the magnetospheric plasmas. In the "cold" plasma approximation, these equations describe toroidal and poloidal shear Alfven waves. The pressure influence leads to the appearance of a new type of oscillations — the magnetosonic waves, which are coupled with the poloidal shear Alfven waves. The stability of the poloidal shear Alfven waves have been analyzed. They are unstable at large pressures.

Initial equations. As a starting point the following set of small oscillation equations

$$\rho\frac{d\vec{\mathbf{v}}}{dt} = -\nabla\tilde{p} + \frac{1}{c}\left(\vec{\mathbf{j}}\times\vec{\mathbf{B}}\right)^{\sim}, \quad \tilde{\vec{\mathbf{B}}} = \mathrm{rot}\left(\vec{\xi}\times\vec{\mathbf{B}}\right), \quad \tilde{p} = -\left(\vec{\xi}\cdot\nabla p\right) - \gamma p\,\mathrm{div}\vec{\xi}, \qquad (1)$$

which describes the arbitrary MHD perturbations of an ideal compressible plasma is being considered. Here the generally accepted notations are used for all physical quantities. The tilde ($^{\sim}$) sign indicates a perturbed component.

After the introduction of the quantities

$$\vec{\mathbf{T}} \equiv \tilde{\vec{\mathbf{B}}} + \frac{\vec{\mathbf{j}}\times\nabla\psi}{|\nabla\psi|^2}\left(\vec{\xi}\cdot\nabla\psi\right) = T_1\nabla\psi + T_2\frac{\vec{\mathbf{B}}\times\nabla\psi}{|\nabla\psi|^2} - T_3\vec{\mathbf{B}}, \quad \vec{\xi} = \xi\frac{\nabla\psi}{|\nabla\psi|^2} + \eta\frac{\vec{\mathbf{B}}\times\nabla\psi}{|\vec{\mathbf{B}}|^2} + \tau\frac{\vec{\mathbf{B}}}{|\vec{\mathbf{B}}|^2},$$

where ψ is the magnetic surface mark (see [1] for details), equations (1) can be rewritten in the following form

$$\frac{1}{|\vec{\mathbf{B}}|^2|\nabla\psi|^2}\rho\frac{\partial^2\xi}{\partial t^2} = \frac{\nabla\psi\cdot\nabla T_3}{|\nabla\psi|^2} + \frac{1}{|\vec{\mathbf{B}}|^2}\left[\frac{\nabla\psi\cdot\nabla T_0}{|\nabla\psi|^2} - 2p'T_3 + T_2\left(S-\gamma_s\right) + \vec{\mathbf{B}}\cdot\nabla T_1 + K\xi\right],$$

$$\frac{|\nabla\psi|^2}{|\vec{\mathbf{B}}|^2}\rho\frac{\partial^2\eta}{\partial t^2} = \frac{\vec{\mathbf{B}}\times\nabla\psi}{|\vec{\mathbf{B}}|^2}\nabla T_0 + \vec{\mathbf{B}}\cdot\nabla T_2 + \left(\vec{\mathbf{B}}\times\nabla\psi\right)\cdot\nabla T_3, \quad \rho\frac{\partial^2\tau}{\partial t^2} = \vec{\mathbf{B}}\cdot\nabla T_0. \qquad (2)$$

Here the expressions for T_i, S, α_s, γ_s, K are given by

$$T_1 = \frac{\vec{\mathbf{B}}\cdot\nabla\xi}{|\nabla\psi|^2}, \quad T_2 = \frac{1}{\alpha_s}\left[\vec{\mathbf{B}}\cdot\nabla\eta + \left(\gamma_s - S\right)\xi\right],$$

$$T_3 = \frac{\nabla\psi\cdot\nabla\xi}{|\nabla\psi|^2} + \frac{\vec{\mathbf{B}}\times\nabla\psi}{|\vec{\mathbf{B}}|^2}\nabla\eta + \frac{\xi}{|\vec{\mathbf{B}}|^2}[\mathrm{div}\left(\alpha_s\nabla\psi\right) + 2p'], \qquad (3)$$

$$S = \frac{\vec{\mathbf{B}}\times\nabla\psi}{|\nabla\psi|^2}\cdot\mathrm{rot}\frac{\vec{\mathbf{B}}\times\nabla\psi}{|\nabla\psi|^2}, \quad \alpha_s = \frac{|\vec{\mathbf{B}}|^2}{|\nabla\psi|^2}, \quad \gamma_s = \frac{\vec{\mathbf{j}}\cdot\vec{\mathbf{B}}}{|\nabla\psi|^2},$$

$$K = \frac{\alpha_s}{\gamma_s}(\gamma_s - S) + \frac{p'}{|\vec{\mathbf{B}}|^2}\frac{\nabla\psi\cdot\nabla\left(2p + |\vec{\mathbf{B}}|^2\right)}{|\nabla\psi|^2}$$

and the prime indicates a derivative with respect to ψ. Equations (2), (3) describe the general structure of MHD perturbations and must agree with the equilibrium plasma configuration.

We will use the dipole model of geomagnetic field induced by the dipole momentum M. The longitudinal current in equilibrium state is considered to be zero.

$$\vec{\mathbf{B}} = \nabla\varphi\times\vec{\mathbf{\Psi}}, \quad |\vec{\mathbf{\Psi}}| = M\frac{\cos^2\vartheta}{r}, \quad \vec{\mathbf{j}}\cdot\vec{\mathbf{B}} = 0.$$

Supposing that $\psi = |\vec{\mathbf{\Psi}}|$, we obtain

$$S = \gamma_s = 0.$$

Reduction of the equations. Equation (2) for ballooning perturbations, when

$$\frac{|\nabla\psi\cdot\nabla X|}{|\nabla\psi|^2}, \frac{|\left(\vec{\mathbf{B}}\times\nabla\psi\right)\cdot\nabla X|}{|\vec{\mathbf{B}}||\nabla\psi|^2} \ggg \frac{X}{R_\oplus}, \frac{|\vec{\mathbf{B}}\cdot\nabla X|}{|\vec{\mathbf{B}}|},$$

where $R_\oplus$ is the Earth radius, X — any component of the plasma shift vector, takes the form

$$\begin{aligned}
&\frac{\omega^2\rho}{|\nabla\psi|^2}\xi + \left(\vec{\mathbf{B}}\cdot\nabla\right)\frac{\left(\vec{\mathbf{B}}\cdot\nabla\right)\xi}{|\nabla\psi|^2} - \frac{2r}{|\vec{\mathbf{M}}|a^2}\left(T_0 + p'\xi\right) = 0,\\
&\frac{\omega^2|\nabla\psi|^2\rho}{|\vec{\mathbf{B}}|^2}\eta + \left(\vec{\mathbf{B}}\cdot\nabla\right)\frac{\left(\vec{\mathbf{B}}\cdot\nabla\right)\eta}{\alpha_s} = 0,\\
&\rho\omega^2\tau + \vec{\mathbf{B}}\cdot\nabla T_0 = 0.
\end{aligned} \tag{4}$$

Here

$$T_0 = \frac{\gamma p|\vec{\mathbf{B}}|^2}{\gamma p + |\vec{\mathbf{B}}|^2}\left(\mathrm{div}\frac{\tau\vec{\mathbf{B}}}{|\vec{\mathbf{B}}|^2} - 2\vec{\kappa}\cdot\vec{\xi}\right), \quad \vec{\kappa} = \frac{\nabla\left(2p + |\vec{\mathbf{B}}|^2\right)}{2|\vec{\mathbf{B}}|^2} - \frac{\vec{\mathbf{B}}}{|\vec{\mathbf{B}}|^4}\left(\vec{\mathbf{B}}\cdot\nabla\right)\frac{|\vec{\mathbf{B}}|^2}{2}.$$

The set (4) describes the coupled poloidal shear Alfven and magnetosonic eigenmodes (the first and the last equations) and the toroidal shear Alfven eigenmodes (the second equation).

Using the dimensionless variables

$$\omega\to\Omega = \frac{\omega}{\omega_A}, \; |\vec{\mathbf{B}}|\to\frac{|\vec{\mathbf{B}}|}{B_0}, \; \psi\to\frac{\psi}{\psi_0}, \quad p\to\frac{p}{B_0^2}, \; B_0 = \frac{M}{L^3}, \; \psi_0 = \frac{M}{L}, \; \omega_A = \frac{B_0}{L\sqrt{\rho}},$$

where L is the McIlwain's parameter, and the relations

$$\left(\vec{\mathbf{B}}\cdot\nabla\right) = \frac{M}{L^4\cos^7\vartheta}\frac{\partial}{\partial\vartheta}, \quad |\nabla\psi| = \frac{M\sqrt{a(\vartheta)}}{L^2\cos^2\vartheta}, \quad a(\vartheta) = 1 + 3\sin^2\vartheta,$$

which describe projection on the field line, it is possible to rewrite (4) in dimensionless form

$$\begin{aligned}
&\Omega^2\xi + \frac{a(\vartheta)}{\cos^{13}\vartheta}\frac{\partial}{\partial\vartheta}\left[\frac{1}{a(\vartheta)\cos\vartheta}\frac{\partial\xi}{\partial\vartheta}\right] - \frac{4}{a(\vartheta)\cos^4\vartheta}\left(T_0 - \frac{\alpha\beta}{\gamma}\xi\right) = 0,\\
&\Omega^2\eta + \frac{1}{\cos^{13}\vartheta}\frac{\partial}{\partial\vartheta}\left[\frac{1}{\cos\vartheta}\frac{\partial\eta}{\partial\vartheta}\right] = 0,\\
&\Omega^2\tau + \frac{1}{\cos^7\vartheta}\frac{\partial T_0}{\partial\vartheta} = 0,
\end{aligned} \tag{5}$$

with

$$T_0 = \frac{\beta|\vec{\mathbf{B}}|^2}{\beta + |\vec{\mathbf{B}}|^2}\left[\frac{1}{\cos^7\vartheta}\frac{\partial}{\partial\vartheta}\left(\frac{\tau\cos^{12}\vartheta}{a(\vartheta)}\right) - \frac{4\cos^2\vartheta}{a^2(\vartheta)}\xi\right], \quad \alpha = -\frac{L}{p}\frac{dp}{dL}, \quad \beta = \frac{\gamma p}{B_0^2}.$$

Set (5) together with boundary conditions

$$\xi, \eta, \tau|_{\vartheta=\pm\vartheta_0} = 0,$$

where "+" and "–" signs correspond to fixed magnetic line ends at the ideally conductive ionosphere, fully describe the spectrum of eigenfrequencies of ballooning modes.

"Cold" plasma approximation. Equations (5) transform into

$$\Omega^2\eta + \frac{1}{\cos^{13}\vartheta}\frac{\partial}{\partial\vartheta}\left[\frac{1}{\cos\vartheta}\frac{\partial\eta}{\partial\vartheta}\right] = 0, \quad \Omega^2\xi + \frac{a(\vartheta)}{\cos^{13}\vartheta}\frac{\partial}{\partial\vartheta}\left[\frac{1}{a(\vartheta)\cos\vartheta}\frac{\partial\xi}{\partial\vartheta}\right] = 0.$$

These equations describe toroidal and poloidal static shear Alfven waves, correspondingly. Their eigenfrequencies are almost the same, except for the lowest modes of both types. For higher-order modes WKB approximation is applicable. With its help the following expression can be obtained for the eigenfrequencies

$$\Omega_n = \frac{\pi n}{\int\limits_{-\vartheta_0}^{+\vartheta_0}\cos^7\vartheta d\vartheta}, \quad n = 1, 2, 3, ...$$

"Warm" plasma approximation. It follows from the energetic concept that the most unstable modes are flute ones, but they cannot exist, because they must be constant along the field line and to disappear on the ionosphere surface due to boundary conditions. The next most unstable are ballooning modes. It follows from the first and the third equations (5), as shown in [2], that these modes are described by equation

$$\Omega^2\xi + \frac{a(\vartheta)}{\cos^{13}\vartheta}\frac{\partial}{\partial\vartheta}\left[\frac{1}{a(\vartheta)\cos\vartheta}\frac{\partial\xi}{\partial\vartheta}\right] + \frac{4\alpha\beta}{\gamma}\frac{\xi}{a(\vartheta)\cos^4\vartheta} - \frac{16\beta}{a(\vartheta)\cos^4\vartheta}\frac{\left\langle\dfrac{\cos^2\vartheta}{a(\vartheta)^2}\xi\right\rangle}{\left\langle 1 + \dfrac{\beta}{a(\vartheta)}\cos^{12}\vartheta\right\rangle} = 0.$$

The stability boundary corresponds to the $\Omega^2 = 0$ condition:

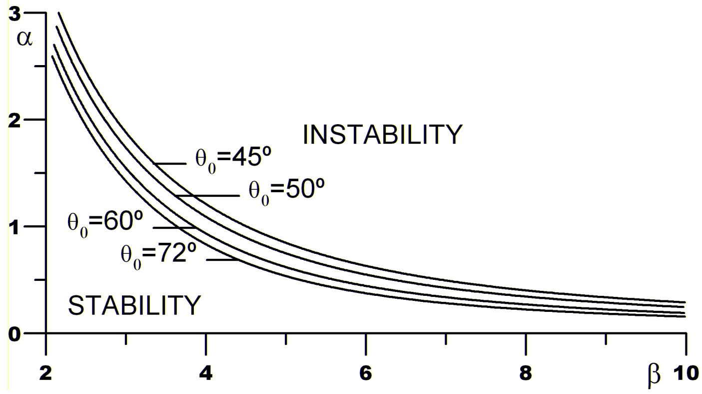

Figure 1: A set of stability boundary curves.

$$\frac{1}{\cos^7\vartheta}\frac{\partial}{\partial\vartheta}\left[\frac{1}{a(\vartheta)\cos\vartheta}\frac{\partial\xi}{\partial\vartheta}\right]+\frac{4\alpha\beta}{\gamma}\frac{\cos^2\vartheta}{a(\vartheta)^2}\xi=\frac{16\beta\cos^2\vartheta}{a}\frac{\left\langle\dfrac{\cos^2\vartheta}{a(\vartheta)^2}\xi\right\rangle}{\left\langle 1+\dfrac{\beta}{a}\cos^{12}\vartheta\right\rangle}.$$

Its solutions are shown on Figure 1.

The main results

- A set of equations (2), describing pressure-driven waves in the dipole magnetic field has been obtained.
- These equations were reduced to a set of scalar partial differential equations (5) for ballooning perturbations.
- The equation of the stability boundary of ballooning modes has been obtained.

References

1. Burdo, O.S., Cheremnykh, O.K., Verkhoglyadova, O.P. Izvestiya RAN, 2000, **64**, N 91, P. 1897 (in Russian).
2. O.K. Cheremnykh, O.S. Burdo, I.A. Kremenetsky, A.S. Parnovskij. To the theory of the MHD waves in the inner magnetosphere of the Earth. Kosmichna Nauka i Technologiya, 2001, **7**, N 5/6, P. 44 – 63 (in Russian).