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# SPECTRAL PROPERTIES OF THE ADVECTION-DIFFUSION EQUATION

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## I. Introduction

The simplest of all transport equations is the advection-diffusion equation characterizing "passive scalar" transport by a specified velocity field. This type of equation has been analyzed over many years by many authors. It continues to form a key core of fluid and plasma transport theory. Perhaps, somewhat surprisingly, there remain important features of such equations which have yet to be fully elucidated.

The purpose of the present paper is to understand and lay bare the essential features of the interaction between flows and transport of some property, such as temperature or density (or, the *turbulent fluctuations* in such quantities). The phenomenon is simplified to its essentials by taking the flow to be *given*, and assigning specific properties such as shear so that the model used captures the key features of more complicated systems. The diffusivity is also assumed to be known. The equation for passive transport is then linear, but turns out to be far from trivial, as one still has to deal with a non-self adjoint, dissipative system. It will be shown that even this simplified model possesses a number of unexpected and remarkable spectral properties which throw light on the evolution of advection-diffusion problems in more complicated (including nonlinear) physical situations [1].

## II. Formulation of the model

We consider a simple two-dimensional problem. Let $\mathcal{L}$ be a two-dimensional domain defined by the rectangle, $0 \leq x \leq L_x, 0 \leq y \leq L_y$. We shall be interested in a scalar function, $f(x, y, t)$ which is supposed to be periodic in $y/L_y$, with period $2\pi$, and which satisfies (for simplicity) homogeneous boundary conditions at $x = 0, x = L_x$. We also suppose that $\nu$ is a uniform and constant diffusivity. It is useful to make contact with tokamak physics and refer to the $x$-direction as "radial" and the $y$-direction as "poloidal". The basic model we start with is represented by the advection-diffusion equation, $\frac{\partial f}{\partial t} + v_y(x)\frac{\partial f}{\partial y} = \nu \left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right]$ , where the advecting velocity, $v_y(x)$ is assumed to be a *given* function of $x$. The equation states that under the influence of $v_y(x)$, the function $f$ (this may represent temperature, magnetic flux, density or any other physical quantity of interest) is *advected* in the poloidal direction, but *diffused* in the radial and poloidal directions. Our fundamental problem therefore consists of elucidating the interplay of the advection and diffusion in this simple system. We note that the parameter of relevance is the Reynolds number, $R = \frac{L_y^2 V'}{\nu}$; hence, it is immaterial physically which parameter is fixed as long as the other varies.

In order to study the properties of this equation, we derive the correspondent eigenvalue problem, $\nu f''(x) + i(\lambda - n v_y(x)) f(x) = 0$ , where the term $\nu \frac{\partial^2 f}{\partial y^2}$ has been ignored because we are only interested in reasonably high Reynolds numbers. This has no general analytic solution for arbitrary velocity profiles, $v_y(x)$, but particular solutions can be found when $v_y(x)$ is a linear, parabolic or sinusoidal function (solutions are, respectively, Airy functions, Hypergeometric functions and Mathieu functions). A simple shooting method code was developed to solve the above equation for arbitrary velocity profiles. In the following section we highlight the results obtained for two interesting examples of $v_y(x)$, i.e., a linear profile and one that is jet-like, that is, zero everywhere except in some small region of the domain, where it assumes a constant, large value.

## III. Discussions and Conclusions

*a) Case of a linear velocity profile*

The eigenvalue spectrum found has the following properties: 1) It is discrete and has a Y shape which is independent of the value of $R$ .This means that no matter how *small the diffusivity*, provided it is non-vanishing, the spectral distribution has an invariant shape, although evidently individual eigenvalues will move about in the complex plane. It is essential to note that the spectral distribution does *not* continuously tend to a subset of the real line (i.e., to the continuous spectrum for the purely advective equation). 2) The density of the eigenvalues increases as $R \to \infty$. 3) as $\nu \to 0$ the eigenfunctions become delta-functions. Points 1 and 2 are illustrated in figure 1.

It is perhaps surprising that a continuous transition in the spectrum from the case $\nu = 0$ (in this case, the spectrum is "continuous" along the portion of the real axis corresponding to the range of $n v_y(x)$, to very small values of $\nu$ (that is, from ideal to dissipative) does not occur. We note that an arbitrarily small amount of diffusivity is sufficient to break the frozen flux constraint, which gives rise to qualitatively very different

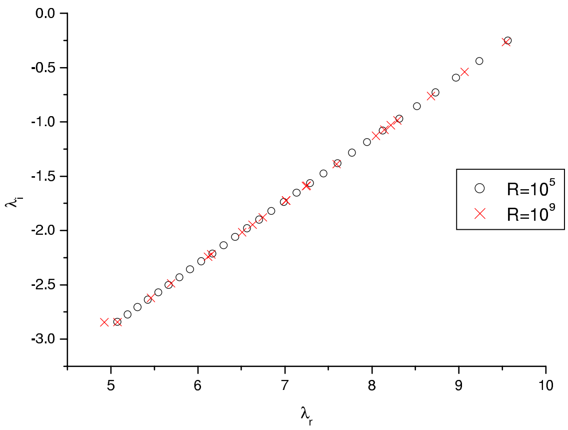


Figure 1: One arm of the "Y" structure for the linear profile spectrum, for two different values of the Reynolds number (both incomplete). Notice the exact overlap of the two structures as well as the density increase in the number of eigenvalues.

behaviour of the system.

This discontinuous transition is not a new result and has been addressed in refs. [2] and [3], in connection with the corresponding phenomenon in ideal MHD. The mathematical explanation for this "paradox" (i.e., the ideal and dissipative spectral distributions have different morphology in the limit as the diffusivity goes to zero) is due to the fact that diffusion is a *parabolic singular perturbation* of the advective (i.e., *hyperbolic*) equation.

*b) Case of a jet velocity profile*

By a "jet" profile we mean a velocity field that is zero everywhere except in a small region of the domain, where it assumes a high value (typically, a Reynolds number $R = 10^4$ is sufficient to validate all the following conclusions). Physically, this kind of profile can be driven by both turbulent Reynolds and Lorentz forces/stresses and corrugations in the ion pressure gradient. In the case of electron physics, current gradients and dynamo effects produce similar changes in advective (i.e., electron inertial) terms. We have studied the effect of a profile composed by two of these jets.

Main results may be summarised as follows: 1) The spectrum has as many branches as the number of regions into which the domain is divided. Thus, for instance, if we are considering the double jet case, the spectrum will have a three arm structure. 2) Different regions of the domain are isolated from each other, implying that eigenfunctions

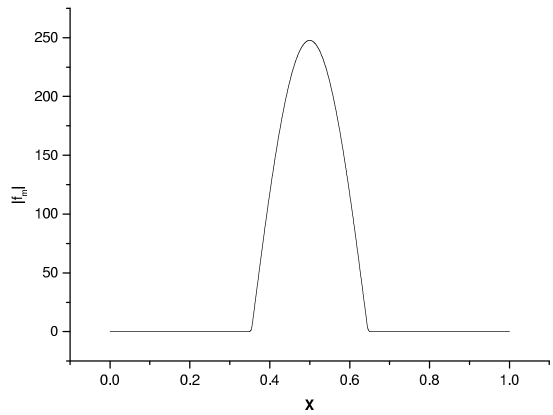


Figure 2: Lowest order eigenfunction (central arm of spectrum) for a double jet profile ($R = 5.10^5$). Complete confinement of the eigenfunction is observed.

will approach zero values near the jets. For high enough Reynolds numbers ($R = 10^5$ is already sufficient) eigenfunctions will only exist in one of the three regions in which the jets divide the domain. We have thus found that this kind of velocity profile confines the eigenfunctions in the regions between the jets. Each branch of the eigenspectrum stands for one of these regions (see figure 2).

The confinement of the eigenfunctions is an interesting result and should be of importance in the limiting of radial correlations and propagation effects associated with the turbulence. Hence, it may alter the turbulent transport. Results are obviously generalizable for an arbitrary number of jets. Finally, we note that suitable comparison with a time-evolution code (CADENCE) again yields very good agreement.

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