Parallel Flows and Plasma Equilibria in Dipolar Magnetic Configurations

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Abstract

The effects of parallel plasma flow on plasma equilibrium in dipolar magnetic configurations (MC) are considered by applying a separable form of magnetic flux function. Both analytical and numerical solutions of the plasma equilibria for sub-Alfvenic plasma flow are presented. However, no physically meaningful solutions with super-Alfvenic plasma flow are found.

I. Introduction. Plasma equilibria in a dipolar MC and the effects of the plasma flows on these equilibria are of interest for both laboratory experiments [1-4] and astrophysical applications [5,6]. The effects of parallel plasma flows on plasma equilibrium in dipolar MC are the topic for this study. From the ideal MHD equations assuming toroidal symmetry we can derive [7] the following equations for the plasma equilibrium in a dipolar MC.

\[ \rho \vec{v} = \vec{B} \Phi(\psi) \] (1)

\[ \vec{B} \cdot \left( \frac{\rho \nabla \vec{v}^2}{2} + \nabla P \right) = 0 \] (2)

\[ \nabla \left( \left(1 - \frac{\Phi^2}{\rho} \right) \frac{1}{R^2} \nabla \psi \right) + \frac{\vec{B}^2}{2\rho} \frac{d\Phi^2}{d\psi} \left| \nabla \psi \right|^2 + \left( \frac{\rho \nabla \vec{v}^2}{2} + \nabla P \right) \nabla \psi = 0 \] (3)

where \( \rho, \vec{v}, \) and \( P \) are the plasma mass density, velocity, and pressure; \( \psi \) is the magnetic flux function; \( \vec{B} = \nabla \psi \times \nabla \Phi \) is the magnetic field; \( \Phi(\psi) \) is the arbitrary function of \( \psi \); and \( R \) is the distance from the major axis. To reduce Eq. (1)-(3) we will follow [8] and introduce a separable form for the flux function \( \psi \) written in spherical coordinates \((r, \varphi, \mu = \cos \theta)\), i.e. the anzatz

\[ \psi(\mu, r) = \psi_0 h(\mu) (r_0 / r)^\alpha, \] (4)

where \( \alpha \) is an adjustable parameter, \( h(\mu) \) is an unknown function of \( \mu \) alone, and \( \psi_0 \) and \( r_0 \) are normalization constants \( h(\mu) \) and \( \alpha \) play the roles of the eigenfunction and eigenvalue of
the nonlinear Grad-Shafranov equation. To apply anzatz (4) we need (see Eq. 3) to assume
\[ \left( \Phi(\varphi) \right)^2 / \rho(\tilde{r}) \equiv \left( \nu/V_A \right)^2 \equiv W(\mu), \tag{5} \]
where \( V_A \) is the Alfvén velocity and \( W(\mu) \) is some function of \( \mu \) alone. Substituting Eq. (5) in (2) and (3) and assuming that \( P=0 \) we find that product \( BW \) supposed to be a function of \( \psi \). Since \( W \) is a function of \( \mu \) alone, using the anzatz (4), which corresponds to
\[ B^2 \propto \left( \alpha^2 \mu^2 / (1 - \mu^2) + (d\psi / d\mu)^2 \right)^{-2(\alpha+2)}, \]
we find the expression for \( W(\mu) \),
\[ W(\mu) = \frac{W_0}{h^{(\alpha+2)/\alpha}} \left( h^2 / (1 - \mu^2) + \alpha^{-2}(d\psi / d\mu)^2 \right)^{1/2}, \tag{6} \]
where \( W_0 \geq 0 \) is a constant, as well as the equation for the function \( h(\mu) \)
\[ \frac{d}{d\mu} \left( 1 - W \right) \frac{dh}{d\mu} + \alpha(\alpha + 1) \frac{(1 - W)h}{1 - \mu^2} + \alpha(\alpha + 2) \frac{W_0^2}{W} h^{(\alpha+4)/\alpha} = 0, \tag{7} \]
which describes equilibrium of pressureless plasma with parallel flow in dipolar MC. As the boundary conditions we can take
\[ h(\mu^2 \rightarrow 1) \propto 1 - \mu^2, \quad h(\mu = 0) = 1, \quad \text{and} \quad (d\psi / d\mu)|_{\mu=0} = 0, \tag{8} \]
which implies, finite magnetic field at the major axis, normalization of the constant \( \psi_0 \) (4), and “up-down” symmetry of the MC. Thus the 2-nd order Eq. (7) having 3 boundary conditions is over-determined and it’s solution is only possible for some special relation between the constants \( W_0 \) and \( \alpha \), \( \alpha = \alpha(W_0) \). From Eq. (7) we need to find both \( \alpha = \alpha(W_0) \) and \( h(\mu) \), corresponding to this \( \alpha \). We will consider only positive \( \alpha \). Then at a given flux surface \( \psi \) we have \( r \propto h(\mu)^{1/\alpha} \bigg|_{\mu \rightarrow \pm 1} \rightarrow 0 \) and the magnetic flux is reminiscent to the MC of a point dipole.

II. Solution. A. The case \( W_0 < 1 \). For \( W_0 \ll 1 \) we find the following dependence \( \alpha = \alpha(W_0) \)
\[ \alpha - 1 \approx -\frac{W_0}{2} \int_0^1 \frac{(1 - \mu^2)^3 (1 + 15 \mu^2)}{(1 + 3 \mu^2)^{3/2}} d\mu \approx -0.49 \times W_0. \tag{9} \]
Eigenfunction \( h(\mu) \) corresponding to the above \( \alpha \) we find from the following expression
\[
\frac{1-f(\mu)}{W_0} \approx \hat{f}(\mu) \equiv \int_0^{1/2} d\mu' \left\{ \frac{2(1-\mu'^2)^2}{(1+3\mu'^2)^{3/2}} + \frac{1}{2} \left[ 2 + \mu' \right] \frac{\int_0^{1/2} d\mu'' \left( \frac{1-\mu''^2}{1-\mu'^2} \right)^2 \left( 1+15\mu''^2 \right)^{1/2}}{(1+3\mu''^2)^{3/2}} \right\}.
\]

(10)

here \( f(\mu) = h(\mu)/(1-\mu^2) \). The function \( \hat{f}(\mu) \) found numerically from (10) is shown in Fig.1. For \( W_0 < 1 \) the equation (7) with the boundary conditions (8) was solved numerically. The dependence \( \alpha(W_0) \) found from these calculations is shown in Fig.2. Note that for \( W_0 < 1 \) the eigenvalue \( \alpha(W_0) \) found from the numerical solution agrees well with analytic estimate (9).

**B. The case \( W_0 > 1 \).** This case could correspond to the super-Alfvenic plasma flow if the boundary conditions (8) would be applicable. But it can be shown analytically that \( W_0 > 1 \) is not compatible with the boundary conditions (8), which describe well-behaved dipolar configuration with finite magnetic field. However we can relax the boundary conditions (8) and consider equilibria with an azimuthal current sheet located at \( \mu = 0 \). In this case the dipolar magnetic field line changes direction when it goes through the current sheet causing \( (dh/d\mu) \bigg|_{\mu=0} \) to be finite and discontinuous. This extra freedom allows both \( h(\mu = 0) = 1 \) and \( h(\mu^2 \to 1) \approx 1 - \mu^2 \) to be satisfied. Therefore, when azimuthal current sheet is allowed at \( \mu = 0 \) the boundary conditions for Eq.(7) become

\[
h(\mu = 0) = 1, (p(\mu = \pm 0))^2 = p_0^2 \equiv W_0^2 - 1 \text{ and } h(\mu^2 \to 1) \approx 1 - \mu^2,
\]

(11)

Notice, that the plasma flow velocity at \( \mu = 0 \) just reaches the Alfven velocity even when \( W_0 > 1 \). For \( W_0 \gg 1 \) the solution of Eq.(7) with boundary conditions (11) can be found analytically. From (7) we see that for \( p^2 \gg 1 \) we have \( dp/d\mu \sim O(1) \). Since \( \mu \) varies from 0 to 1 we can neglect the variation of \( p \) and assume that \( p^2 \approx p_0^2 \gg 1 \). Then from (7) we find that to satisfy (11) we should have \( \alpha \approx 1/W_0 \) and \( h(\mu) \approx 1 - |\mu| \). Notice, that flux surfaces corresponding to this solution are squeezed about equatorial plane. We also solve Eqs. (7),(11) numerically to determine the function \( \alpha(W_0) \) which is shown in Fig.3. We find that
\( \alpha \) varies continuously as \( W_0 \) passes through unity and both \( \alpha(W_0) \) and \( h(\mu) \) approach the limit obtained analytically at large \( W_0 \).

**III. Conclusions.** We find both analytical and numerical solutions of the equation describing plasma equilibrium in dipolar MC for sub-Alfvénic plasma flow and no physically meaningful solutions with super-Alfvenic plasma flow. We conclude that there is no steady-state axisymmetric equilibrium permitting super-Alfvénic parallel plasma flow for the class of separable dipolar magnetic configurations considered.

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**References**


**Figures**

**Fig. 1** The function \( \tilde{f}(\mu) \) found from numerical integration of expression (10).

**Fig. 2** \( \alpha(W_0) \) found from numerical solution of Eq. (7) and (8) for \( W_0 < 1 \).

**Fig. 3** \( \alpha(W_0) \) found from numerical solution of Eq.(7) and (11) for \( W_0 > 1 \).