Turbulence self-modulation and poloidal asymmetry of the tokamak plasma fluxes

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The turbulence sustained by the gradient of the ion temperature (ITG turbulence) is considered as one of the main sources of the transport which limits the energy confinement in tokamaks. It is known that the differential equation describing the ITG mode has turbulent as well as very regular exact solutions of the vortex type. This makes very problematic the use of renormalization methods, since the later are necessarily perturbative. New methods have been proposed [1], inspired by nonperturbative (“semiclassical”) methods developed in field theory. A problem of great interest is the relation between the ITG turbulence and the sheared poloidal plasma rotation, due to the strong effect of the rotation on the radial correlation length of the fluctuating field with possible formation of transport barriers. Beside the Reynolds stress as a source of rotation torque and inverse spectral cascade (sustained by the ion polarization nonlinearity) the possibility has been noticed that the ITG turbulence generates the zonal flows via the modulation instability of the envelope of the fluctuating field [2]. The studies have been focused on the possibility that the modulational instability provides very long poloidal wavelength needed to build the zonal flow.

We present here a complementary point of view on the effects of the modulational instability. Our object of analysis is the ITG turbulent electrostatic field in the presence of sheared plasma rotation. The main objective is to find the possible evolutions of the envelope of the amplitudes of the fluctuating field, analysing their stability properties. The main conclusion is that the ITG-turbulent plasma spontaneously evolves to a poloidally asymmetric profile of the turbulence amplitude; since this induces a transport rate which is poloidally nonuniform, a torque arises which acts to enhance the rotation of the plasma.

Our results are the following:

- We derive a single equation describing the ITG turbulence in the presence of the sheared poloidal rotation and find that it is of the barotropic form, well studied in the fluid and atmospheric sciences.

- We find that the self-modulation can be described by a set of two coupled Cubic Nonlinear Schrödinger Equations (CNSE) for the amplitudes of two propagating waves representing the envelope of the turbulent field on slowertime and longer-space scales.

- We provide several solutions of this system representing slowly propagating solitons or, in physical terms, poloidal deformations which constitutes an asymmetric distribution of the turbulence amplitude in the poloidal direction. It appears quite clear that the poloidally asymmetric turbulence distributions arise spontaneously from the dynamics of the ITG.
Since periodic space-uniform and space-nonuniform (poloidally symmetric and respectively asymmetric) solutions are equally possible, we study which of them will hold in tokamak plasma. For this we carry out a detailed analysis of the spectral properties of the exact solutions of the integrable CNSE, using the geometric-algebraic framework of the Inverse Scattering Transform on periodic domains \(i.e.,\) poloidal circumference). We find that the space-uniform solution (also called plane wave) is unstable and any perturbation to the initial conditions induces a separation from this one at an exponential rate. A solution providing a nonuniform envelope (turbulence level with poloidal asymmetry) is necessarily established.

Since plasma evolves spontaneously to a poloidally asymmetric profile of the fluctuating field, the transport rate will also present poloidal asymmetry. We invoke the Stringer mechanism of the poloidal plasma spin-up to pose the problem of the effectiveness of the torque associated with the self-modulation. We compare the damping due to the neoclassical magnetic pumping with the force associated with the poloidally varying rate of transport. Since the latter is exponentially growing in time when a perturbation is applied on the uniform solution, the associated torque can overcome the damping.

Deriving the nonlinear equation for the ITG field we assume neutrality and adiabaticity. We replace the electron and ion velocities into the continuity and momentum equations. For electrons, \(v_e = \frac{-\nabla \times \tilde{n}}{B} + \frac{T_e}{|e| B n_0} \frac{1}{dr} \tilde{e}_y\) and for ions

\[
\mathbf{v}_{\perp,i} = \frac{T_i}{|e| B n_0} \frac{1}{dr} \tilde{e}_y - \frac{1}{B \Omega_i} \frac{d}{dt} (\nabla_{\perp} \phi) + \frac{-\nabla \phi \times \tilde{n}}{B}
\]

After neglecting small terms according to the ITG ordering, we obtain:

\[
\beta \frac{\partial \phi}{\partial x} + \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla_{\perp}^2 \phi - \frac{d^2 U}{dy^2} \frac{\partial \phi}{\partial x} + \varepsilon \left( -\nabla_{\perp} \phi \times \tilde{n} \right) \cdot \nabla_{\perp} \nabla_{\perp}^2 \phi = 0
\]

where \(U(y)\) is the sheared poloidal velocity having as typical value \(U_0\) and \(\beta = \Omega_i \frac{T_i}{|e| B n_0} \frac{1}{dr} \tilde{e}_y\) and \(\varepsilon = \frac{1}{T_i B n_0} \frac{1}{dr} \tilde{e}_y\) are dimensionless constants. \(x = r \theta\) is the poloidal direction, \(d = 2 \pi a\) and the rest of notation is standard. We perform a multiple time and space scales analysis, \(T_1 = \varepsilon t, T_2 = \varepsilon^2 t\) and \(X_1 = \varepsilon x, X_2 = \varepsilon^2 x\), with \(\phi = \phi^{(1)} + \varepsilon \phi^{(2)} + \varepsilon^2 \phi^{(3)} + \cdots\). The result is

\[
i \frac{\partial A}{\partial T} + \frac{\partial^2 A}{\partial X^2} + |A|^2 A + \tilde{v}_{12} |B|^2 A = 0, \quad i \frac{\partial B}{\partial T} + \alpha \frac{\partial^2 B}{\partial X^2} + |B|^2 B + \tilde{v}_{12} |A|^2 B = 0
\]

where \(A\) and \(B\) are amplitudes of the waves which constitute the envelope function \(\psi^{(2)}\). This system is the coupled CNSE. Many solutions are known and are classified by group-theoretical methods: travelling solitons, breathers, etc. all of them consisting of localized perturbations of finite width on the space variable. A typical solution is

\[
B(x,t) = \sqrt{\frac{2}{\alpha + 1}} \sec h \left( \sqrt{\frac{2(\alpha - 1)}{\alpha + 1}} x \right) \exp (i t)
\]
with scaled variables \((x, t)\). This already shows that the turbulence envelope supports solutions which are not uniform in the angle \(\theta\).

The non-trivial space-uniform solution must be time dependent: \(\phi(x, t) = s \exp(2i s^2 t)\). We have examined the stability properties of the space-uniform solution in the framework of the geometric-algebraic approach to the integrable CNSE on the periodic spatial domain \([3]\). We give in the following a brief description of the method. Since the Lax operator \(L\) is known for CNSE we can construct the matrix of fundamental solutions of the equation \(L \phi = \lambda \phi\). Calculated at the periodic boundary \((x_0, x_0 + d)\) this is the monodromy matrix \(M\), giving the automorphisms of the space of solutions. Using the fundamental solutions we can construct any other solution and we look for the Bloch functions, whose monodromy is represented via the the Floquet exponents, \(p\): \(\psi(x_0 + d) = \exp[ip(\lambda)] \psi(x_0)\). These exponents govern the stability properties of the Bloch functions and they are obtained as eigenvalues of the monodromy matrix: \(m^2 - \Delta(\lambda)m + 1 = 0\), where \(m = \exp[ip\theta]\) and \(\Delta(\lambda) = Tr(M)\). In different regions of the complex \(\lambda\) plane we have different stability properties, and in particular \(m = \pm 1\) for the points of the main spectrum \(\Delta^2(\lambda) = 2\). If \(\lambda_j\) is a point in the main spectrum, to which correspond two independent Bloch function, the eigenvalue is said to be degenerate. Otherwise (one eigenvalue \(\to\) one Bloch function) it is said nondegenerate. The nondegenerate eigenvalues are the zeros of the Wronskian (since the Bloch functions are identical). The square of the Wronskian is a polynomial in \(\lambda\) which can be represented as a two-sheeted branched covering of the complex \(\lambda\) plane, \(\therefore\) a genus \(N\) hyperelliptic Riemann surface (where \(2N\) is the number of conjugated nondegenerate eigenvalues). The geometry of this surface is the essential factor leading to the exact solution of the CNSE (via differential forms, cycles and periods, Abel map and Jacobi inversion using the theta functions). The stability properties of the solution \(\phi(x, t) = s \exp(2i s^2 t)\) is governed by this surface: any perturbation with a small function \(\varepsilon h(x)\) breaks the degenerate eigenvalues and from their splitting arise new branching points. The handles of the Riemann surface proliferate and new degrees of freedom becomes active. This is the origin of the instability of the space-uniform envelope of the ITG turbulence. A slight perturbation of this solution induces a departure from the symmetric state and a poloidally asymmetric solution is developing.

The Lax eigenvalue problem for the plane wave solution and the fundamental matrix are
\[
\begin{pmatrix}
  i \frac{\partial}{\partial x} & 1 \\
  -1 & -i \frac{\partial}{\partial x}
\end{pmatrix}
\Phi = \lambda \Phi, \quad \Phi(x; \lambda) = \begin{pmatrix}
  \cos k x - \lambda \frac{i \sin k x}{k} \\
  i \frac{\sin k x}{k} & \cos k x + \frac{i \sin k x}{k}
\end{pmatrix}
\]
The monodromy matrix is \(M(\lambda) = \Phi(x = d; \lambda)\) and the discriminant \(\Delta(\lambda) = Tr M(\lambda) = 2 \cos [d \sqrt{1 + \lambda^2}]\). The main spectrum is \(\lambda_n^0 = \pm (n^2 \pi^2 / d^2 - 1)^{1/2}\) corresponding to \(\Delta(\lambda) = 2\). A small perturbation induces the \(\varepsilon\)-splitting of the degenerate eigenvalue \(\lambda_n^0\). It has been shown that only the degenerate eigenvalues with finite imaginary part produce unstable behaviour of the exact solution. In particular we remark that for fixed \(d\) there is a finite number of purely imaginary eigenvalues. For example \(\pm i\), but the latter are nondegenerate and are connected by a cut in the \(\lambda\) plane. The splitting of a degenerate \(\lambda_n^0\) produces a new handle, two new cycles and
four new periods. The new geometry of the Riemann surface imposes new values for the the integrals of the differential forms over the cycles and these integrals are the arguments of the theta functions which provide the solution (Jacobi inversion). When the arguments are calculated taking into account the perturbation \( \varepsilon \) it results a wavenumber \( k_j = -2\sqrt{1 + (\lambda^0)^2} \) and frequency \( \Omega_j = \pm k_j \sqrt{1 + (\lambda^0)^2} + O(\varepsilon) \). For \( \varepsilon \to 0 \) these values verify the linear dispersion relation of CNSE. When the wavelength of a perturbation is \( k < 2s \) there is an exponential growth with a rate \( \gamma = \text{Im}\Omega = |k\sqrt{k^2 - 4s^2}| \). The instability of the uniform solution is a functional phenomenon but we can roughly say that the growth of the nonsymmetric solution is exponential with the rate \( \gamma \).

The poloidally asymmetric distribution of the turbulence yields a correspondingly asymmetric radial transport rate. This in turn provides a poloidal torque via the Stringer mechanism. A poloidal force \( R_\theta \) is related to a radial flux \( nV_r = -1/(eB_T) R_\theta \) and this velocity has the surface average \( V_r = \langle V_r \rangle \) and a surface varying part \( \tilde{v}_r = \langle 2\cos \theta V_r \rangle \). For a diffusion which is \( \theta \)-dependent \( V_r = D_{\text{ITG}}/L_n \) where \( D_{\text{ITG}} = D_{\text{ITG}} w(\theta, t) \) is the basic ITG diffusion coefficient modulated with the envelope represented by \( w(\theta, t) = \sec h^2(r\theta/d) \exp(2\gamma t) \). The momentum balance for the poloidal plasma rotation must take into account the damping due to the neoclassical magnetic pumping. The equation is [4]:

\[
\Theta \left( 1 + 2q^2 \right) \left( \frac{\partial V_\theta}{\partial t} + \gamma_{\text{MP}} V_\theta \right) + qV_\theta \frac{1}{nr} \frac{\partial}{\partial r} \left( nr\tilde{v}_r \right) = 0
\]

with \( \gamma_{\text{MP}} = \frac{2}{2} \left( \frac{B_T}{B_0} \right)^2 \frac{r}{1+\nu} \left( \frac{\nu L_{\text{th}}}{c_s} \right), \Theta = r/(qR_0), \nu \) the ion collision frequency, \( \epsilon = r/R_0 \). \( V_\theta \) must be bounded from below by \( U_0 \) which was assumed initially. Denoting by \( \delta \) the radial width of the seed sheared velocity the time required for the asymmetry-induced poloidal force to become greater than the magnetic pumping is \( t \sim \frac{1}{2\gamma} \ln \left[ \frac{\gamma_{\text{MP}}}{(2\cos \theta \sec h^2(\theta_0/d)) \frac{\delta}{q D_{\text{ITG}} c_s}} \right] \). This means that \( t \) is of the same order as the \( s^{-2} \) in physical units. This will increase the poloidal rotation. In conclusion, the poloidal asymmetry of the turbulence is an intrinsic property of the ITG model and renders unstable a sheared poloidal rotation.

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References


