Multi-mode Parametric Excitation of Bernstein Waves under Electron Cyclotron Heating

A.A. Ivanov\(^1\), K.S. Serebrennikov\(^1\), V.Yu. Fedotov\(^1\), M. Bacal\(^2\), J.-M. Buzzi\(^2\).

\(^1\)Russian Research Center “Kurchatov Institute”, Moscow, Russia
\(^2\)Laboratoire de Physique des Milieux Ionises, UMR 7648 du C.N.R.S., Ecole Polytechnique, 91128 Palaiseau Cedex, France

If a homogeneous plasma undergoes an influence of a potential electric field \(\vec{E} = \vec{E}_0 \sin(\omega_0 t)\) perpendicular to an external magnetic field \(\vec{B}\), then an instability is excited in the plasma. One can represent the time development of the instability as Floque series

\[ \varphi = \exp(i\omega t) \sum a_n \exp(i\omega_n t) \]

here \(\varphi\) is a potential of the excited potential waves [1].

Substituting the series in Vlasov’s equation we obtain an infinite system of algebraic equations in \(a_n\). The solvability condition for the system gives the following equation for \(\omega\)

\[ D(\omega, \vec{k}) = \begin{bmatrix} I & D^e \\ D^i & I \end{bmatrix} = 0 \]

here \(D(\omega, \vec{k})\) is a block matrix, \(I\) is the unit matrix, \(D^e\) and \(D^i\) are blocks with the following elements: \(D^e_{mn} = R_e^{(n)} J_{n-m}(\mu)\), \(D^i_{mn} = R_i^{(n)} J_{m-n}(\mu)\); \(\mu = |\vec{k}|^2\), \(\vec{k}\) is the wave vector of the excited wave, \(\vec{b}\) is the amplitude of electron oscillations, \(\Omega\) is the electron gyrofrequency,

\[ R^{(n)}_a = \frac{\delta e_a (\omega + n\omega_0, \vec{k})}{1 + \delta e_a (\omega + n\omega_0, \vec{k})}, \quad a = e, i. \]

If \(\vec{k} \perp \vec{B}\) and one put ions to be nonmagnetized then

\[ \delta e_e = -\frac{\omega^2_{\text{pi}}}{\omega^2} \quad \text{and} \quad \delta e_i = -2 \sum_{n=1}^{\infty} \frac{n^2 \Omega^2 \Phi_n (\vec{k})}{n^2 - \omega^2 - \Omega^2} \]

here \(\Phi_n = I_n (k^2 \rho_e^2 \exp(-k^2 \rho_i^2)) / k^2 \rho_e^2\), \(\omega_{\text{pi}}\) is the ion plasma frequency, \(r_{\text{De}}\) and \(\rho_e\) are the electron Debye and gyro radii, \(I_n\) is the modified Bessel function.

Having followed [2] and [3] we shall simplify equation (1). One can easily prove that the infinite determinant \(D(\omega, \vec{k})\), regarded as a function of complex variable \(\omega\), appears to be even and periodic with a period equal to \(\omega_0\). Elements of the matrix \(D(\omega, \vec{k})\) have no singularities, but for poles. The poles are roots of the equation \(1 + \delta e_a (\omega, \vec{k}) = 0\). Then according to Mittag-Leffler’s theorem \(D(\omega, \vec{k})\) can be represented as follows

\[ D(\omega, \vec{k}) = N(\omega) + \sum_u \sum_n \frac{K_{un}}{n^2 (\omega / \omega_0)^n - \sin^2 (\pi \omega_n / \omega_0)}, \]

where \(N(\omega)\) is an entire periodic function, \(K_{un}\) are the
poles of $D(\omega, \tilde{k})$, and the coefficients $D_{an}$ are obtained by substituting $\omega_{an}$ in $D(\omega, \tilde{k})$ with subsequent regularization of the string containing the pole $\omega_{an}$. Easy test shows that $N(\infty) = 1$ and by Liouville’s theorem $N(\omega) = 1$

Therefore analytical properties of the determinant $D(\omega, \tilde{k})$ permit us to express it in terms of simple periodic functions of $\omega$ and coefficients $D_{an}$ independent on $\omega$. Taking into consideration that $\omega_{pi} / \omega_0 << 1$ we obtain up to the order of $(\omega_{pi} / \omega_0)^3$:

$$D(\omega, \tilde{k}) = 1 + \frac{\pi \omega_{pi}}{2 \omega_0} \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^{\infty} \frac{J_{n+\ell}^* (\mu) J_{n-\ell}^* (\mu)}{\sin(\pi \lambda^-)} - \frac{J_{n+\ell} (\mu) J_{n-\ell} (\mu)}{\sin(\pi \lambda^+)}$$

(1)

where $\lambda^\pm = \frac{\omega \pm \omega_{pi}}{\omega_0}$.

This formula is valid for arbitrary values of $\omega$.

1. Dispersion equation for arbitrary wave length

Denote $\omega = i \gamma$, $K = k \rho_e$, $\Delta = \omega_0 - \Omega$ and $R = \rho_e / \rho_p = \Omega / \omega_{pe}$, $a = \frac{1}{2} \frac{E_0}{B} \nu_T e$

and let $R >> 1$, $\omega / \Omega << 1$, and $\delta = \Delta / \Omega << 1$.

Then the positive roots of the equation $1 + \delta_c (\omega, K) = 0$ are

$$\omega_{en} = n \Omega (1 + \Phi_n (K))$$

Now expression (1) can be simplified, and we obtain the following dispersion equation

$$1 - \frac{\omega_{pi}^2}{\gamma^2 + \omega_{pi}^2} \sum_{n=1}^{\infty} \frac{2 n \Omega (\Omega \Phi_n - \Delta) \Phi_n \lambda^2 J_n^2 (\mu)}{\gamma^2 + \omega_{pi}^2} = 0$$

(2)

Here $\mu = \frac{a}{\delta} K$.

2. Dispersion equations for long and short wave lengths

If $K >> 1$ (short wave lengths) then (2) can be reduced to

$$1 - \frac{\omega_{pi}^2}{\gamma^2 + \omega_{pi}^2} \Phi(K) \left( 1 - \frac{\pi \nu}{\sin \pi \nu} J_n (\mu) J_n (\mu) \right) = 0$$

(3)

Here $\nu = \gamma / (\Omega \Phi - \Delta)$.

This equation contains contribution of all of the Bernstein modes. We search positive solutions of equation (3), i.e. $\gamma > 0$.

If $K << 1$ (long wave lengths) then $\Phi_n / \Phi_1 << 1$, and we keep in (2) the terms with $n = 1$ only.

$$1 - \frac{\omega_{pi}^2}{\gamma^2 + \omega_{pi}^2} 2 \Omega^2 J_1^2 (\mu) \frac{(\Phi_1 (K) - \delta) \Phi_1 (K)}{\gamma^2 + \Omega^2 (\Phi_1 (K) - \delta)^2} = 0$$

(4)

3. Strong electric fields

Strong electric fields are characterized by the inequality

$$a > \Phi_1 (0) = \frac{1}{2 R^2} \text{ i.e. } e E_0 > m v_T e \frac{\omega_{pe}^2}{\Omega}$$

The range of values of $\delta$ is split up into two physically different domains. The first is

$$0.3 \Phi_1 (0) = \delta_1 < \delta < \delta_0 = \Phi_1 (0) \text{ i.e. } 0.15 \frac{\omega_{pe}^2}{\Omega} < \Delta < \frac{1}{2} \frac{\omega_{pe}^2}{\Omega}$$

(5)

In this range long waves ($K = k \rho_e < 1$) are excited.

The second domain is
In this range short waves ($K = k \rho_e > 1$) are excited.

Given $\delta$ external field excites waves with various values of wave vector. We shall search wave vectors with maximal linear growth rate $\gamma$, i.e. we must find from (3) and (4) the linear growth rate and solve the equation $\partial \gamma / \partial K = 0$.

### 3.1 Long waves

For the case of long waves (equation (4) and inequality (5)) we have the following expression for the maximal growth rate $\gamma_m$

$$\gamma_m^2 = -\frac{1}{2} \Omega^2 (\Phi_1(0) - \delta)^2 - \frac{1}{2} \omega_{pi}^2 + \frac{1}{2} \sqrt{\left[ \Omega^2 (\Phi_1(0) - \delta)^2 + \omega_{pi}^2 \right]^2 + 4 \Omega^2 \omega_{pi}^2 (\Phi_1(0) - \delta)(\delta - 0.3 \Phi_1(0))}$$

The wave vector corresponding to the maximal growth rate is

$$K_m = 1.84 \frac{\delta}{a} \ll 1 \quad \hbar_m = 1.84$$

$\gamma_m$ as a function of $\delta$ has a maximum. So the growth rate as function of $K$ and $\delta$ has the absolute maximum $\Gamma$.

$$\Gamma = 0.55 \omega_{pi} \left( \frac{\omega_{pe}^2}{\omega_{ti}^2} \right)^{1/3}$$

This value is achieved at the following values of $\delta$ and $K$

$$\delta = \tilde{\delta} = \frac{1}{2} \frac{\omega_{pe}^2}{\Omega^2} - 0.55 \frac{\omega_{pi}}{\Omega} \left( \frac{\omega_{pe}^2}{\omega_{ti}^2} \right)^{1/3} \quad \text{and} \quad K = \tilde{K} = 1.84 \frac{\tilde{\delta}}{a}$$

### 3.2 Short waves

For the case of short waves (equation (3) and inequality (6)) the maximal growth rate and corresponding wave vector have the form

$$\gamma_m = \frac{2}{3} \omega_{pi} \left( \frac{a \Omega}{\omega_{pi}} \right)^{1/3} \left( \frac{\omega_{pe}^2}{\sqrt{2} \pi \Omega^2} \right)^{1/9}$$

and

$$K_m = \left( \frac{\omega_{pe}^2}{\sqrt{2} \pi \Omega^2} \frac{1}{\delta} \right)^{1/3} - \frac{1}{3} \left( \frac{0.7 \omega_{pi}}{a \Omega} \right)^{2/3} \left( \frac{\omega_{pe}^2}{\sqrt{2} \pi \Omega^2} \frac{1}{\delta} \right)^{1/9}$$

Substitution of the maximal growth rate (7) in equation (2) shows that approximately $aK_m / \delta$ summands make equal contribution to the dispersion equation and, therefore, a lot of Bernstein modes are excited.

### 4. Weak electric fields

Now consider weak electric field

$$a \ll \Phi_1(0) = \frac{1}{2R^2} \quad \text{i.e.} \quad eE_0 \ll mv_{Te} \frac{\omega_{pe}^2}{\Omega}$$

In this case the range of values of $\delta$ is split up into three domains

$$a^{3/4} (\Phi_1(0))^{1/4} = \delta_1 < \delta < \delta_0 < \Phi_1(0) \quad \text{i.e.} \quad a^{3/4} \left( \frac{\omega_{pe} \Omega}{\sqrt{2}} \right)^{1/2} < \Delta < \frac{1}{2} \frac{\omega_{pe}^2}{\Omega}$$

$$\frac{a^3 \Omega^4}{\omega_{pi}^3} \Phi_1(0) = \delta_2 < \delta_1 \quad \text{i.e.} \quad a^3 \frac{\Omega^4}{\omega_{pi}^3} < \Delta < a^{3/4} \left( \frac{\omega_{pe} \Omega}{\sqrt{2}} \right)^{1/2}$$
\[ \delta < \delta_2 \text{ i.e. } \Delta < \frac{a^3 \Omega^2 \omega_{pe}^2}{2\omega_{pi}^3} \]  

(10)

The growth rate achieves the maximal value \( \gamma_m \) at the following value of \( K \)

\[ K_m = \left( \frac{\omega_{pe}^2}{\sqrt{2\pi\Omega^2} \delta} \right)^{1/3} \left( \frac{\omega_{pi}}{a\Omega} \right)^{2/3} \left( \frac{\omega_{pe}^2}{\sqrt{2\pi\Omega^2} \delta} \right)^{1/9} \]  

for region (8) and (10)

\[ K_m = \left( \frac{a\omega_{pe}^2}{\omega_{pi} \Omega \sqrt{2\pi\delta}} \right)^{1/2} \]  

for region (9)

Corresponding values of \( \gamma_m \) are as follows

\[ \gamma_m = \frac{a^2 K_m \Omega}{4\delta} \]  

for region (8)

\[ \gamma_m = aK_m \Omega \]  

for region (9)

\[ \gamma_m = \frac{2}{3} \omega_{pi} \left( \frac{a\Omega}{\omega_{pi}} \right)^{1/3} \left( \frac{\omega_{pe}^2}{\sqrt{2\pi\Omega^2} \delta} \right)^{1/9} \]  

for region (10)

**Conclusion**

The above account shows that the linear growth rate of parametric excitation of the Bernstein modes may sufficiently exceed \( \omega_{pi} \) even when the external pumping field is weak. The growth rate and the wave vector of the excited waves increases when \( \delta (= \Delta / \Omega) \) decreases.

It should be observed that the growth rate of parametrically excited waves propagating nearly parallel to external magnetic field is less than \( \omega_{pi} \) [4,5]. Therefore the Bernstein modes make a dominant contribution to development of the parametric instability in the vicinity of the gyro frequency.

The most interesting case is one of strong pumping fields and small \( \delta \). In this case short waves are excited and growth rate is more then \( \omega_{pi} \). But unlike weak fields a lot of Bernstein modes are excited simultaneously.

In conclusion notice an important feature of the mathematical formalism employed. Unlike many others authors we were not restricting ourselves to the condition \( \mu < 1 \). This condition corresponds to very long waves. Really, even if \( k_{pe} \mu < 1 \) the maximal growth rate is achieved at \( \mu = 2 \) (see section 3.1).

**References**