31st EPS Conference on Plasma Phys. London, 28 June - 2 July 2004 ECA Vol.**28G**, P-5.191 (2004)

Second stability regime of the ideal internal kink mode in a tokamak

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Stability theory of the $m = n = 1$ mode in toroidal plasmas provides the basis for understanding the sawtooth oscillations in tokamaks. One of the most important contributions in this area of tokamak physics is the pioneering work by Bussac *et al.* [1] on the ideal magnetohydrodynamic (MHD) stability of the internal kink mode in a toroidal plasma with large aspect ratio and circular cross section. In terms of Δq and β_p, where $\Delta q = 1 - q_0$, q is the safety factor, q_0 its value at the magnetic axis, and β_p the poloidal beta value at the $q = 1$ radius $r = r_1$, an approximate, normalized form of the potential energy δW of the ideal $m = n = 1$ mode derived by Bussac *et al.* is given by

$$\delta\hat{W} = \delta\hat{W}_{\text{Bussac}} = \Delta q\left(\frac{13}{48} - 3\beta_p^2\right). \tag{1}$$

This expression is valid for a parabolic current profile near the axis, and for small values of r_1 and Δq. Furthermore, β_p is assumed to be of order unity in Eq. (1). The stability condition $\delta\hat{W}_{\text{Bussac}} > 0$ then leads to the well-known pressure limit $\beta_p < \beta_{p,crit} = \sqrt{13}/12 \approx 0.3$ for stability of the ideal, internal kink mode in a toroidal plasma. This pressure limit, as well as the more general result obtained from the complete expression for δW calculated by Bussac *et al.* (valid for finite Δq and r_1, and leading to values of $\beta_{p,crit}$ in the range 0.1-0.3, depending on the current profile) agrees with numerical results when $\varepsilon_a << 1$, where $\varepsilon_a = a/R_0$, a is the minor radius and R_0 the major radius of the plasma. The agreement appears to be rather poor, however, for values of ε_a relevant for most tokamaks ($\varepsilon_a > 0.2$) [2, 3]. Furthermore, it was found in numerical studies by Tokuda *et al.* [4] and by Manickam [5] that the ideal $m = n = 1$ mode has a second stability regime when β_p is of order unity. This is the case also when Δq is very small, and is not predicted by Eq. (1), in spite of the fact that the expansion parameter $\varepsilon_1 \equiv r_1/R_0$ of the calculation leading to Eq. (1) goes to zero as $\Delta q \to 0$.

It is suggested in the present work that a probable explanation for the inaccuracy of the Bussac theory when ε_a is finite is that the second term in the large aspect ratio expansion of δW starting with Eq. (1) is of the same order of magnitude as the first term in the parameter regime where Eq. (1) is valid. Noting that the potential energy δW of the mode is of order ε_1^2 [although the factor ε_1^2 has been normalized away in $\delta\hat{W}$ in Eq. (1)], the expansion of $\delta\hat{W}$ including also the next-order term can be written

$$\delta\hat{W} = \delta\hat{W}^{(2)} + \varepsilon_1^2 \delta\hat{W}^{(4)} + \ldots , \qquad (2)$$

where the superscript denotes the order with respect to ε_1 and $\delta\hat{W}^{(2)} \equiv \delta\hat{W}_{\text{Bussac}}$. If both $\delta\hat{W}^{(2)}$ and $\delta\hat{W}^{(4)}$ are of order unity in Eq. (2), the second term is negligible since ε_1^2 in practice is a very small number, of the order of 10^{-2} or smaller. However, since $\delta\hat{W}^{(2)}$ happens to be proportional to Δq when Δq is small, we should compare ε_1^2 with Δq rather than with unity in order to estimate the relative importance of the two terms in Eq. (2), provided that $\delta\hat{W}^{(4)} \sim 1$ when Δq is small. Furthermore, since $\varepsilon_1^2 = (r_1 / R_0)^2 = (r_1 / a)^2 \varepsilon_a^2$ we see that, if ε_a is finite, Δq should be compared with $(r_1 / a)^2$ instead. It is easy to show from a cylindrical calculation that, for an axial current distribution of the form $J(r) = J_0 (1 - r^2 / a^2)^\nu$, $(r_1 / a)^2 \approx 2\Delta q / \nu$ if Δq and r_1 are small, indicating that the two terms in Eq. (2) should be of similar magnitude when Δq is small, ε_a finite and the current profile parabolic near the axis. For such parameters it is thus necessary to extend the Bussac theory with the second term in Eq. (2) in order to obtain an asymptotically correct expression for the potential energy of the ideal $m = n = 1$ mode.

In order to calculate the second term in Eq. (2), we employ the expansion method developed in Ref. 6 for the calculation of δW for the ideal $m = n = 1$ mode in toroidal plasmas. The poloidal side-bands generated in an expansion of the ideal MHD equations up to fourth order in ε_1 are given by:

$O(1)$:	1				
$O(\varepsilon_1)$:	0	2			
$O(\varepsilon_1^2)$:	– 1	1	3		
$O(\varepsilon_1^3)$:	– 2	0	2	4	
$O(\varepsilon_1^4)$:	– 3	– 1	1	3	5

The numbers in boldface denote the side-bands that have to be taken into account in a calculation of the $m = 1$ component of δW to order ε_1^4. Out of the six poloidal side-bands $\neq 1$ above, $m = 0$ and 2 to order ε_1 are present already in the Bussac theory. Furthermore, $m = -1$ and 3 to order ε_1^2 were calculated in the analysis of the ellipticity effect in Ref. 6, and this result can be used directly here. It remains to calculate the $m = 0$ and $m = 2$ side-bands to order ε_1^3, as well as the final $m = 1$ component to order ε_1^4. The details of this calculation can be found in Ref. 7, and here we only summarize the main structure of the $m = 1$ equation and the final result of the analysis.

The marginal m = 1 amplitude ξ_1 is found to be determined by the following equation, including all the $O(1)$, $O(\varepsilon_1^2)$ and $O(\varepsilon_1^4)$ terms and side-band couplings:

$$\frac{d}{dr}\left[r^3(\mu-1)^2\frac{d\xi_1}{dr}\right]+W_1\xi_1+\frac{d}{dr}\left(r^3W_2\frac{d\xi_1}{dr}\right)+r^2W_3\frac{d\xi_1}{dr}+\frac{d}{dr}\left(r^3W_4\frac{d\xi_2}{dr}\right)+\frac{d}{dr}\left(r^2W_5\xi_2\right)$$

$$+\sum_{m=-1}^{3}N_m\xi_m=0, \tag{3}$$

where $\mu\equiv 1/q$. The operators N_m above are defined by

$$N_mf\equiv\frac{d}{dr}\left(r^3W_{m1}\frac{df}{dr}\right)+\frac{d}{dr}\left(r^2W_{m2}f\right)+r^2W_{m3}\frac{df}{dr}+rW_{m4}f\ , \tag{4}$$

and the coefficient functions W_m can be found in Ref. 7. The terms $N_m\xi_m$ in Eq. (3) represent the new, fourth-order terms not included in the Bussac theory [1], and describe couplings to the side-bands m = −1 up to m = 3. These side-bands are, in turn, driven by the m = 1 amplitude, and some of them also couple to each other. The equations for the side-band amplitudes ξ_m are also given in Ref. 7. The expression for the potential energy δW is then obtained by integration of Eq. (3) from r = 0 to $r = r_1 - 0$:

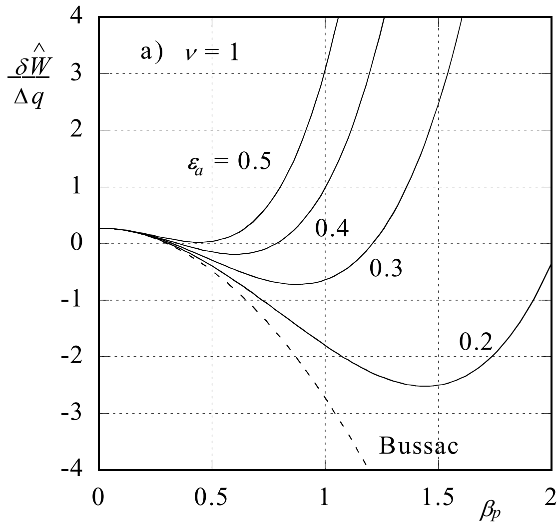
$$\hat{\xi}\delta W^{(2)}=-\hat{\xi}\int_0^{r_1-0}W_1dr-r^3W_4\left.\frac{d\xi_2^{(1)}}{dr}\right|_{r=r_1-0}-r^2W_5\xi_2^{(1)}\Big|_{r=r_1-0}, \tag{5}$$

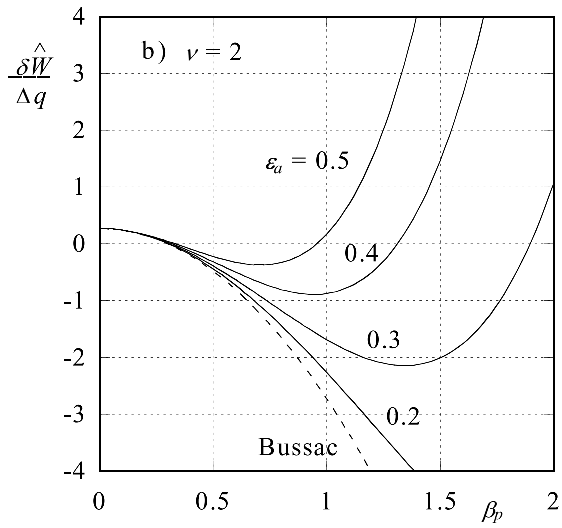
$$\hat{\xi}\delta W^{(4)}=-\int_0^{r_1-0}\left(\sum_{m=-1}^{3}N_m\xi_m\right)dr-r^3W_4\left.\frac{d\xi_2^{(3)}}{dr}\right|_{r=r_1-0}-r^2W_5\xi_2^{(3)}\Big|_{r=r_1-0}. \tag{6}$$

$\delta\hat{W}^{(2)}$ and $\delta\hat{W}^{(4)}$ in Eq. (2) are thereafter given by $\delta\hat{W}^{(2)}=R_0^2\delta W^{(2)}/r_1^4$ and $\delta\hat{W}^{(4)}=R_0^4\delta W^{(4)}/r_1^6$, respectively. In Eqs. (5) and (6), $\xi_2^{(1)}$ and $\xi_2^{(3)}$ denote the first- and third-order components of the m = 2 amplitude, respectively, and $\hat{\xi}$ denotes the amplitude of the zeroth-order part of ξ_1, given by $\xi_1^{(0)}=const=\hat{\xi}$ for $r < r_1$, and $\xi_1^{(0)}\equiv 0$ for $r > r_1$. For a parabolic current profile near the axis, and when Δq and r_1 both are small, Eq. (5) leads to the expression for $\delta\hat{W}^{(2)}$ in Eq. (1). By a similar, but much longer, calculation, the following analytical expression for $\delta\hat{W}^{(4)}$ is obtained from Eq. (6) [7]:

$$\delta\hat{W}^{(4)}=6\beta_p^3\left(\beta_p+1\right)+\frac{\beta_p\left(1-14\beta_p\right)}{32}. \tag{7}$$

The quantity $\delta\hat{W}^{(4)}$ in Eq. (7) is positive for all positive values of β_p. Furthermore, $\delta\hat{W}^{(4)}$ is small when β_p is small, but increases rapidly with increasing β_p . This is illustrated in Figs. a and b below, where $\delta\hat{W}/\Delta q$ according to Eq. (2), with $\varepsilon_1^2 = (r_1/a)^2\varepsilon_a^2 \approx 2\varepsilon_a^2\Delta q/\nu$, is shown as a function of β_p for different values of ε_a, and for $\nu = 1$ (a) and for $\nu = 2$ (b) [ν is a profile parameter in the current density $J(r) = J_0(1-r^2/a^2)^\nu$]:





It is seen that, for values of ε_a relevant for most tokamaks ($\varepsilon_a > 0.2$), the effect of $\delta\hat{W}^{(4)}$ in Eq. (2) is very strong for $\beta_p > 0.3$-0.5, and a second stability regime of the $m = n = 1$ mode is obtained already for $\beta_p \sim 1$. This is in agreement with the numerical results in Refs. 4 and 5. It is also consistent with earlier stability theory of high-β plasmas in Refs. 8 and 9. Notice the strong effect of the current profile, where small ν gives better stability than large ν. This has to do with the larger $q = 1$ radius that is obtained for a flat current (for a given Δq), and the effect of the factor $\varepsilon_1^2 \equiv (r_1/R_0)^2$ in Eq. (2).

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