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Stability of Karman vortex streets in flute mode turbulence

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I. In the present paper we consider the stability of large scale vortex structures that can spontaneously be excited by interchange (flute) mode turbulence in magnetized plasma. The flute modes are defined to be low frequency, $\omega << \Omega_{ci}$ (the ion cyclotron frequency), drift-type fluctuations driven by the plasma pressure gradient perpendicular to the ambient magnetic field B. In a slab geometry with profile gradients in the x direction and magnetic field in the z direction these gradient-specific modes are known to be two-dimensional vortex motions which are able to propagate in the y-direction, i.e. the direction of translation symmetry. It is well known that such plasma motions exhibit dual turbulent cascades. Namely, as the wave number spectrum broadens through nonlinear dynamics, the energy is transferred primarily to larger scales. Furthermore, it is becoming more evident now that these same nonlinear dynamics create long-lived, strongly anisotropic large-scale vortex structures. We focus the discussion on the two-dimensional (2-D) vortex flows that are described by the 2-D stationary propagating solutions to the model equations that governs the linear and nonlinear evolution of flute modes. These solutions are "breather" and Kelvin-Stuart "cat's eyes" that are localized in the direction of plasma inhomogeneity, periodic in the direction of the translation symmetry and correspond, physically, to so-called "Karman vortex streets" known in the 2-D fluid dynamics.

II. We suppose that a weakly inhomogeneous plasma of slab geometry is immersed in a constant magnetic field $\mathrm{B}_0 = (0,0,B_0)$. The curvature of magnetic field lines is imitated by the fictional gravity $\mathrm{g} = -\nabla V(x)$. To describe the flute modes in such plasma we use two-fluid macroscopic equations and assume the plasma to be quite dense.
Restricting our consideration to the low-β (ratio of plasma pressure to magnetic field pressure) case, we can use the collisionless electrostatic limit, $\mathrm{E} = -\nabla\phi$, for the low frequency oscillations and neglect the perturbations of the magnetic field. Assuming now that the fluctuation level is small in the sense that ratios $\delta n / n_0$ and $e\phi / T$ are small and the space variation of the fluctuating quantities is more rapid than the corresponding

unperturbed quantities, the basic equations are reduced to a pair of coupled nonlinear equations for dimensionless density $n(\mathrm{r},t) = \delta n / n_0$ and potential $\Phi(\mathrm{r},t) = e\phi / T$,

$$\frac{\partial n}{\partial t} + \mathrm{v}^* \frac{\partial \Phi}{\partial y} = -[\Phi, n] \tag{1}$$

$$\left(\frac{\partial}{\partial t}\nabla^2\Phi - \mathrm{v}^* \frac{\partial}{\partial y}\nabla^2\Phi\right) + \mathrm{v}_0 \frac{\partial n}{\partial y} = -\left([\Phi, \nabla^2\Phi] + \nabla\cdot[n, \nabla\Phi]\right) \tag{2}$$

Here $\mathrm{v}^* = \kappa c T / e B_0 \mathrm{v}_{th}$ and $\mathrm{v}_0 = g / \Omega_{ci} \mathrm{v}_{th}$ are the ion diamagnetic and the gravitational drift velocities normalized to the ion thermal velocity, v_{th}, respectively, $\kappa = -(d \ln n_0 / dx) > 0$, and $g = T / m_i R$, R is the characteristic scale length of magnetic field inhomogeneity. Moreover, we introduce the dimensionless space $\mathrm{r} \to \mathrm{r} / \rho_i$ and time $t \to t / \Omega_{ci}$ variables, where $\rho_i = \mathrm{v}_{th} / \Omega_{ci}$ is the ion Larmor radius.

III. We will show that Eqs.(1) and (2) can be put in the noncanonical Hamiltonian form. To this end we define the new variable $q = n - \mathrm{v}^* x$. Then the energy integral of the system takes the form

$$H = \int \left[\Phi \nabla^2 \Phi + \frac{\mathrm{v}_0}{\mathrm{v}^*} (q + \mathrm{v}^* x)^2 \right] dxdy$$

If now we introduce the state vector

$$\mathrm{u} = \begin{pmatrix} q + \mathrm{v}^* x \\ \nabla^2 \Phi \end{pmatrix}$$

then it is easy to see that Eqs.(1)and (2) can be presented in the Hamiltonian form

$$\frac{\partial \mathrm{u}}{\partial t} = J \cdot \frac{\delta H}{\delta \mathrm{u}}$$

where the functional H naturally plays role of the Hamiltonian, and

$$J = \begin{pmatrix} 0 & [q, \cdot] \\ 1 * \; q, \cdot & \left([\nabla^2 \cdot, q] - [\cdot, \nabla^2 \Phi] + [\nabla \cdot, \nabla (q + \mathrm{v}^* x)]\right) \end{pmatrix}$$

is the noncanonical Poisson matrix. We use this Hamiltonian form to construct integral invariants that arise from the properties of the Lie-Poisson brackets and that do not depend on the specific form of the Hamiltonian. These are the so-called Casimir invariants which satisfy relation $J \cdot \frac{\delta C}{\delta \mathrm{u}} = 0$. This matrix equation has the solution

$$C = \int \left[F(q) + \left\{ \nabla^2 \Phi + \frac{1}{2} \nabla^2 (q + v^* x) \right\} q \right] dx dy$$

where *F(q)* is an arbitrary function. These constants are Casimir invariants. Furthermore, since Hamiltonian contains a continuous symmetry in the *y* coordinate we can use Noether's theorem to find the remaining integral of the basic equations given by

$$M = -\int x \nabla^2 \Phi dx dy$$

Here *M* is interpreted as conserved momentum.

IV. Consider stationary solutions of Eqs.(1) and (2) which propagate with constant velocity $u\hat{y}$. Setting $\partial / \partial t = -u \partial / \partial y$ and introducing the streamfunction $\psi = \Phi - ux$ reduces these equations to

$$\nabla^2 \psi = r(\psi) + 0{,}5 g x$$

which is a relation between the streamfunction and the vorticity, here $r(\psi)$ is an arbitrary function. We consider two possible cases, namely, $r_1(\psi) = A e^{-\psi / A}$, and $r_2(\psi) = 2B \sinh \psi$ corresponding to the stationary solutions to the Liouville equation and to the "sinh-Poisson equation" respectively. Under some restriction on free parameters these solutions describe, physically, so-called "Karman vortex streets". These are the *Kelvin-Stuart cat eyes* solution and the "*breather*" solution. The *Kelvin-Stuart cat eyes* solution is determined by

$$\psi_1 = 2A \ln \left[2a \cosh \frac{x}{\sqrt{8}} + 2\sqrt{(a^2 - 1)} \cos \frac{y}{\sqrt{8}} \right] + \frac{g}{12} x^3$$

where $a > 1$. The parameter *a* describes the width of the Kelvin-Stuart cat eyes. As *a* decreases to 1, the cat's eyes diminish and the limiting flow is purely zonal. The "*breather*" solution is given by

$$\psi_2 = 4 \operatorname{arctanh} \left(\frac{b}{a} \frac{\sin ay}{\cosh bx} \right) + \frac{g}{12} x^3$$

where $B = b^2 - a^2$ and $a, b > 0$. As can be seen these solutions describe vortex flow $\left(\nabla^2 \psi \neq 0 \right)$ which is localized in the *x*-direction and periodic in *y*. Moreover, the solutions are non-symmetric in the x-direction.

V. We use the Lyapunov's direct method to investigate the stability of obtained stationary solutions with respect to small perturbations (linear analysis). To this end we construct a Lyapunov functional *L* by means of the integral invariants,

$$L = -H + C + \lambda M =$$
$$\int \left[-\frac{1}{2}\Phi\nabla^2\Phi - \frac{g}{2\kappa}(q+\kappa x)^2 + F(q) + \left(\nabla^2\Phi + \frac{1}{2}\nabla^2(q+\kappa x) \right) q + \lambda x \nabla^2\Phi \right] dxdy$$

where λ is a Lagrange multiplier. The first variation of the Lyapunov functional L is indeed can be equal to zero, when n and $\nabla^2\Phi$ vary arbitrarily and independently. The second variation of L should be of definite sign to ensure stability. It is

$$\delta^2 L = \int \left\{ \left| \nabla\delta\Phi - \nabla\delta q \right|^2 - 2\left[\left| \delta q \right|^2 + r'(q)(\delta q)^2 \right] \right\} dxdy > 0$$

If $r' > 0$ nothing can be concluded about stability from (21) unless a specific relation between δq and $\delta\nabla^2\Phi$ is assumed. On the other hand, if $r' < 0$ we can estimate an upper bound on the linear perturbation wave number, k_0, for which we have stability. Then, the stability criterion $\delta^2 L > 0$ takes the form

$$k_0^2 < 8b^2 \left(2a\cosh\left(\frac{x_{\max}}{\sqrt{8}} \right) + 2\sqrt{a^2 - 1} \right)^{-2}$$

and

$$k_0^2 < |B|$$

for the Kelvin-Stuart cat eyes and "breather" solutions respectively. Thus, linear stability of stationary solutions describing vortex flows for long wavelength perturbations is proved.

VI. We have identified the Hamiltonian structure of the model equations in terms of non-canonical Poisson brackets and used this to find a complete set of integral invariants Two-dimensional stationary non-linear solutions of the model equations in the form of localized structures which may be the result of self-organization in the interchange mode turbulence are obtained. We have considered two particular classes of these solutions, the Kelvin-Stuart cat eyes and "breather", known in 2-D fluid dynamics as vortex streets. To investigate the stability of obtained solutions, the Lyapunov's direct method was used and a Lyapunov functional was constructed from the full set of invariants. By varying this functional we found that the vortex street solutions are linearly stable to long wavelength perturbations.